ABSTRACT SPACES AND LINEAR OPERATORS

METRIC SPACES

$\mathcal{M} = (X, d)$ — X set, d distance

$d : X \times X \to \mathbb{R}$ is a distance if

(i) $d(x,y) = 0 \iff x = y$

(ii) $d(x,y) \leq d(x,z) + d(y,z)$

$\Rightarrow$ (iii) $d(x,y) = d(y,x)$ (from (ii) with $x = z$ we have $d(z,y) \leq d(y,z)$ $\Rightarrow$ given the arbitrarity of y and z we must have $d(y,z) = d(z,y)$)

$\Rightarrow$ (iv) $d(x,y) \geq 0$ (from (ii) with $x = y$ and (i))

$\mathcal{M}(X,d)$ is a metric space if d is a distance

Note that $(X, d_1) \neq (X, d_2)$ if $d_1 \neq d_2$!

example $X = \mathbb{R}^n$ $\quad X = \{x \mid x = (x_1, x_2 \dots x_n)\}$ $\quad x_i \in \mathbb{R}$

$d_1(x,y) = \left(\sum_i (x_i - y_i)^2\right)^{1/2}$ $\qquad d_2(x,y) = \max_i |x_i - y_i|$ $\qquad \mathcal{M}_1 = (X, d_1) \neq \mathcal{M}_2 = (X, d_2)$

NOTE THAT: Physicists call "metric" what is indeed not a metric!

4 vectors $x^\mu = (t, \underline{x})$ $\qquad d(x,y') = x^\mu x'^\nu g_{\mu\nu}$ $\qquad g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$d(x,x') = tt' - \underline{x} \cdot \underline{x}'$ $\qquad d(x,x') = 0 \not\Rightarrow x = x'$!!

Given a metric space we can extend various concepts seen in Analysis I

$\mathcal{M} = (X, d)$ $\qquad$ "ball" around $x_0 \in X$, $r \in \mathbb{R}^+$ $\qquad B(x_0, r) = \{x \in X \mid d(x, x_0) < r\}$ "OPEN"

$\overline{B}(x_0, r) = \{x \in X \mid d(x, x_0) \leq r\}$ "CLOSED"

Neighbourhood : $B(x_0, \varepsilon)$

- **OPEN** $Y \subset X$ is open if $\forall y \in Y \ \exists \varepsilon > 0 \mid B(y, \varepsilon) \subset Y$

- **CLOSED** $Y \subset X$ is closed if its complement is open (complement: $X - Y$)

  Note that $Y \subset X$ can be neither closed nor open!

EXAMPLE

$\mathcal{M} = (\mathbb{R}, d) \qquad d(x,y) = |x-y|$

ϕ is open and closed

$[0,1]$ is closed

$[0,1)$ is neither open nor closed

$(0,1)$ is open

- **CONVERGENCE** $x_i \in X$ $\quad \lim_{n \to \infty} x_n = x$ if $\forall \varepsilon > 0 \ \exists N(\varepsilon) \mid n > N$
  $\Rightarrow d(x_n, x) < \varepsilon$

- **CONTINUITY** $T: \mathcal{M}_X \to \mathcal{M}_Y \qquad \mathcal{M}_X = (X, d_1) \quad \mathcal{M}_Y = (Y, d_2)$

  T is continuous in x_0 if $\forall \varepsilon > 0 \ \exists \delta(\varepsilon) \mid d_1(x, x_0) < \delta$
  $\Rightarrow d_2(f(x), f(x_0)) < \varepsilon$

- **CAUCHY SEQUENCE** $x_i \in X \qquad \mathcal{M} = (X, d)$

  $\{x_i\}$ is a Cauchy sequence if $\forall \varepsilon > 0 \ \exists N(\varepsilon) \mid n, m > N \Rightarrow d(x_n, x_m) < \varepsilon$

- **COMPLETENESS**

  $\mathcal{M}(X,d)$ is COMPLETE if every Cauchy sequence converges in $\mathcal{M}$

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Examples

$$M_1 = (C[a,b], d_1)$$

$$M_2 = (C[a,b], d_2)$$

$C[a,b]$ continuous functions over $[a,b]$

$$d_1 = \max_{t\in[a,b]} |x(t) - y(t)| \qquad d_2 = \int_a^b |x(t) - y(t)|\, dt$$

- M_1 is complete. Indeed let's take a Cauchy sequence $\{x_i\}$

$$\forall \varepsilon>0 \ \exists N(\varepsilon) \ \Big| \ \max_{t\in[a,b]} |x_n(t) - x_m(t)| < \varepsilon$$

$\Rightarrow$ since $\mathbb{R}$ is complete at a given t $x_n(t)$ converges in $[a,b]$ uniformly to $x(t)$

Is $x(t)$ continuous in t_0 ?

Let us write

$$|x(t) - x(t_0)| \leq |x(t) - x_m(t)| + |x_m(t) - x_m(t_0)| + |x_m(t_0) - x(t_0)|$$

$\Rightarrow$ $\forall \varepsilon>0 \ \exists \delta>0, \exists N(\varepsilon) \ \Big| \ \begin{matrix} |t-t_0|<\delta \\ m > N(\varepsilon)\end{matrix} \Rightarrow$

$|x(t) - x_m(t)| < \frac{\varepsilon}{3}$ because x_m converges in t

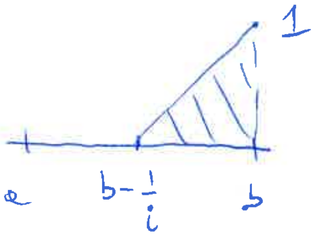
$|x_m(t) - x_m(t_0)| < \frac{\varepsilon}{3}$ because $x_m \in C[a,b]$

$|x_m(t_0) - x(t_0)| < \frac{\varepsilon}{3}$ because x_m converges in t_0

$\Rightarrow$ $|x(t) - x(t_0)| < \varepsilon$ $\Rightarrow$ $x(t)$ is continuous and thus belongs to $C[a,b]$

- M_2 is <u>NOT</u> complete

$$x_i = \begin{cases} 0 & 0 \leq t \leq b - \frac{1}{i} \\ it + 1 - ib & b - \frac{1}{i} \leq t \leq 1 \end{cases}$$



$$d_2(x_n, x_m) \to 0 \qquad d_2(x,y) = \int_a^b |x(t) - y(t)|\, dt$$

$$x_i(t) \to x(t) = \begin{cases} 0 & 0 \leq t < 1 \\ 1 & t = 1 \end{cases} \qquad x(t) \notin C[a,b] \ !$$

4)

EXAMPLE

$\mathbb{Q}$ set of rational numbers with $d(x,y) = |x-y|$ is NOT complete

$x_0 = 1 \quad x_1 = \frac{14}{10} \quad x_2 = \frac{141}{100} \quad \dots \quad x_n = \frac{[10^n \sqrt{2}]}{10^n} \rightarrow \sqrt{2} \notin \mathbb{Q}$

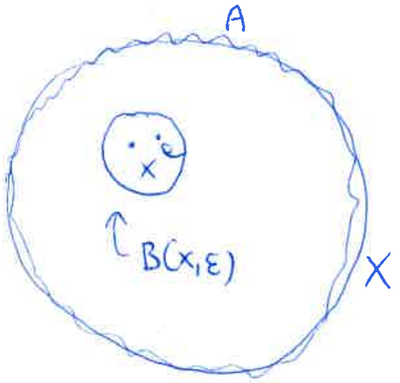
$[x]$ = largest integer in x

$\Rightarrow \mathbb{R}$ is the COMPLETION of $\mathbb{Q}$

Given (X,d) not complete metric space, a completion is a pair $((X',d'),\phi)$ where (X',d') is a complete metric space and ϕ is an isometry between (X,d) and (X',d') | $\phi(X)$ is dense* in X'

Isometry: a mapping $\phi : (X,d) \rightarrow (X',d')$ | $d(x_1,x_2) = d'(\phi x_1, \phi x_2)$

* $A \subset X$ is DENSE in X if $\forall x \in X$ either $x \in A$ or every neighbourhood of x contains at least a point of A.



Example: $\mathbb{Q}$ is dense in $\mathbb{R}$, because every real number can be approximated as well as we want by a rational number

THEOREM (without proof)

Given (X,d) not complete metric space, a completion always exists and it is UNIQUE (up to isometries)

BANACH SPACES

Up to now we have considered X as an arbitrary set: from now on X is a VECTOR SPACE on a field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$)

We define a NORM as a mapping $\|\cdot\| : X \to \mathbb{R}$ (also if $\mathbb{K} = \mathbb{C}$) that fulfills the following properties:

(i) $\|\alpha \underline{u}\| = |\alpha| \, \|\underline{u}\| \qquad \underline{u} \in X \quad \alpha \in \mathbb{K}$

(ii) $\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$

(iii) $\|\underline{u}\| = 0 \implies \underline{u} = \underline{0}$

(iv) $\|\underline{0}\| = 0$ (because $\underline{0} = 0\,\underline{0}$ and $|0| = 0$ so (iii) also holds in the opposite sense

(v) $\|\underline{u}\| \geq 0$ $\left(0 = \|\underline{u} - \underline{u}\| \leq \|\underline{u}\| + \|-\underline{u}\| = 2\|\underline{u}\| \right)$

Every NORM induces a METRIC $d(x, y) = \|x - y\|$

PROOF

(i) $d(\underline{x}, \underline{y}) = 0 \iff \underline{x} = \underline{y}$

$\underline{x} = \underline{y} \implies d(\underline{x}, \underline{y}) = \|\underline{0}\| = 0$

$d(\underline{x}, \underline{y}) = 0 \implies \|\underline{x} - \underline{y}\| = 0 \implies \underline{x} = \underline{y}$

(ii)

$d(\underline{x}, \underline{y}) = \|\underline{x} - \underline{y}\| = \|\underline{x} - \underline{z} + \underline{z} - \underline{y}\| \leq \|\underline{x} - \underline{z}\| + \|\underline{z} - \underline{y}\| = d(x, z) + d(z, y)$

$\implies$ EVERY NORMED SPACE IS A METRIC SPACE, WITH THE METRIC INDUCED BY THE NORM

5)

A **Banach space** is a normed vector space which is **complete** with respect to the metric induced by the norm

example $C[a,b]$ continuous functions over $[a,b]$ $\quad x(t) \in C[a,b]$

1 $\quad \|x\| = \max_{t\in[a,b]} |x(t)|$ → complete → Banach space! (see previous example)

2 $\quad \|x\| = \int_a^b |x(t)|\,dt$ → not complete → **it's not a Banach space!**

3) LINEAR OPERATORS

$$T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$$

X, Y normed vector spaces over the **same** field ($\mathbb{R}$ or $\mathbb{C}$)

$x_1, x_2 \in X \qquad T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$

D_T domain $\{x \in X \mid Tx \text{ is defined}\}$

R_T range $\{y \in Y \mid y = Tx\}$

N_T kernel $\{x \in X \mid Tx = 0\}$

null element in Y

example $\quad T : (C[a,b], \|\cdot\|) \to (K, \|\cdot\|)$ linear functionals (e.g. calculus of variations)

space of test functions

$T : (C_c^\infty[\mathbb{R}^n], \|\cdot\|) \to (K, \|\cdot\|)$ **DISTRIBUTIONS**

$C_c^\infty(\mathbb{R}^n)$ class C^∞ functions with compact support

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BOUNDED LINEAR OPERATORS

$T : X \to Y$ X, Y normed spaces T linear

def: T is **BOUNDED** if $\exists c \in \mathbb{R} \mid \|Tx\| \leq c\|x\| \quad \forall x \in D_T$

If this is the case the minimum c is $\sup\limits_{\substack{x \in D_T \\ x \neq 0}} \dfrac{\|Tx\|}{\|x\|} = \sup\limits_{\substack{x \in D_T \\ \|x\|=1}} \|Tx\| \equiv \|T\|$

this step is due to the linearity!

(set $x = \|x\| y$ $\Rightarrow$ $\|x\|$ cancels in the ratio)

$\Rightarrow \|Tx\| \leq \|T\| \, \|x\|$

THEOREM

Let X, Y be normed vector spaces and let

$T : X \to Y$ a linear operator

The following statements are equivalent

(i) T is bounded

(ii) T is continuous on X

(iii) T is continuous in $x_0 \in X$

PROOF

(i) $\Rightarrow$ (ii)   let us choose $x_0 \in X$ and $\varepsilon > 0$ $\Rightarrow$ choose $\delta = \varepsilon / \|T\|$

$\forall x \mid \|x - x_0\| < \delta$ we have $\|Tx - Tx_0\| = \|T(x - x_0)\| \leq \|T\| \, \|x - x_0\| < \varepsilon$

$\Rightarrow$ T is continuous

(ii) $\Rightarrow$ (iii)   is trivial

(iii) $\Rightarrow$ (i)   Suppose T is continuous in x_0 $\Rightarrow \forall \varepsilon > 0 \; \exists \delta > 0 \mid \|x - x_0\| < \delta \Rightarrow \|Tx - Tx_0\| < \varepsilon$

Choose $y \neq 0$ in D_T and set $x = x_0 + \dfrac{\delta}{\|y\|} y$ $\Rightarrow$ $x - x_0 = \dfrac{\delta}{\|y\|} y$

$\Rightarrow \|Tx - Tx_0\| = \|T(x - x_0)\| = \left\| T \dfrac{\delta}{\|y\|} y \right\| = \dfrac{\delta}{\|y\|} \|Ty\| < \varepsilon \Rightarrow \|Ty\| < \dfrac{\varepsilon}{\delta} \|y\|$ ∎

BOUNDED!

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HILBERT SPACES

We consider a vector space X over a field k $(=\mathbb{R}$ or $\mathbb{C})$

$x, y, z \in X \qquad \alpha, \beta \in k$

We define a scalar product (inner product) on X as a mapping from $X \times X \to k$

with the following properties:

$(x, y) \in k$

$\langle x, y\rangle$ DIRAC NOTATION

(i) $\langle x | \alpha y + \beta z\rangle = \alpha \langle x|y\rangle + \beta \langle x|z\rangle$

(ii) $\langle x|y\rangle = \langle y|x\rangle^*$

(iii) $\langle x|x\rangle \geq 0$ and $\langle x|x\rangle = 0 \iff x = 0$

NOTE: sometimes the linearity is imposed on the first argument!

NOTE: With this definition the scalar product is ANTILINEAR on the first argument

$\langle \alpha x | y\rangle = \langle y | \alpha x\rangle^* = \alpha^* \langle x|y\rangle$

NOTE: if $k = \mathbb{R} \Rightarrow \langle x|y\rangle = \langle y|x\rangle$

Given a definition of scalar product $\Rightarrow$ a NORM is induced as $\|x\| = \sqrt{\langle x|x\rangle}$

Let us check that this is indeed a norm

$$\|ax\|^2 = \langle ax | ax\rangle = |a|^2 \|x\|^2 \quad \text{ok}$$

$$\|x+y\|^2 = \langle x+y | x+y\rangle = \|x\|^2 + \|y\|^2 + \langle x|y\rangle + \langle y|x\rangle$$

$$= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\langle x|y\rangle$$

$$\leq \|x\|^2 + \|y\|^2 + 2\,|\langle x|y\rangle|$$

$$\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2 \quad \text{ok}$$

THIS LAST STEP IS DUE TO THE CAUCHY-SCHWARZ INEQUALITY (to be proved later)

$\|x\| = \langle x|x\rangle = 0 \Rightarrow x = 0$ ok

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A vector space with a scalar product is also called as PRE-HILBERT SPACE

Since the scalar product induces a NORM, and the norm induces a METRIC, we can establish if the space is COMPLETE

$\Rightarrow$ $\boxed{\text{A HILBERT SPACE IS A VECTOR SPACE WITH SCALAR PRODUCT WHICH IS COMPLETE WITH RESPECT TO THE METRIC INDUCED BY THE NORM}}$

- Parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

$\Rightarrow$ if a norm vector space does not respect this rule it cannot be a Hilbert space!

- Cauchy-Schwarz inequality

$$|\langle x|y\rangle| \leq \|x\|\,\|y\|$$

$x=0$ or $y=0$ obvious

$\Rightarrow$ $x \neq 0, y \neq 0$

$$x = \frac{\langle y|x\rangle}{\|y\|^2}\, y + z \qquad \cancel{\langle y|x\rangle} \pm \cancel{\langle y|x\rangle} + \langle y|z\rangle$$

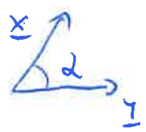
$\Rightarrow$ $\langle y|z\rangle = 0$

$$\Rightarrow \|x\|^2 = \frac{|\langle y|x\rangle|^2}{\|y\|^4}\|y\|^2 + \|z\|^2 \geq \frac{|\langle y|x\rangle|^2}{\|y\|^2} \qquad \blacksquare$$

example

$X = \mathbb{R}^2$
$= (x_1, x_2)$

$$|\langle x|y\rangle| = |x_1 y_1 + x_2 y_2| = \|x\|\,\|y\|\,|\cos\alpha|$$
$$\leq \|x\|\,\|y\|$$



EXAMPLE $\mathbb{C}^m$ $x = (x_1 \dots x_m)$ $x_i \in \mathbb{C}$

$$\langle x|y\rangle = \sum_{i=1}^{m} x_i^* y_i$$ is a Hilbert space

(i), (ii) and (iii) easily checked

EXAMPLE ℓ^2 $x = (x_1 \dots\dots)$ such that $\sum_{i=1}^{\infty} |x_i|^2 < \infty$

$$\langle x|y\rangle = \sum_{i=1}^{\infty} x_i^* y_i \qquad \|x\| = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$$

note that the sum converges (follows from CS inequality at finite m)

EXAMPLE

$C[a,b]$ we have already considered this space as a normed space; is it a HS?

- $\|x\|_1 = \max_{t \in [a,b]} |x(t)|$ we have already seen that this is a Banach space

  but $\|x\|_1$ does not fulfill the parallelogram rule

  $\Rightarrow$ it does not come from a scalar product $\Rightarrow$ NOT A HS

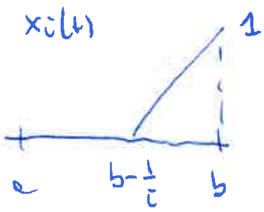
- $\|x\|_3 = \left[\int_a^b |x(t)|^2 dt\right]^{1/2}$ this comes from a scalar product

  $\Rightarrow$ it fulfills the parallelogram rule

$$\langle x|y\rangle = \int_a^b dt\, x^*(t)\, y(t)$$

However $C[a,b]$ with this norm is NOT COMPLETE

$\Rightarrow$ THIS IS NOT A HS (pre Hilbert space)



x_i are Cauchy sequence but $x_i \to x$ $\begin{cases} x(t) = 0 & a \le t < b \\ x(t) = 1 & t = b \end{cases}$

$x(t) \notin C[a,b]$

EXAMPLE $L^2[a,b]$ functions defined over $[a,b]$ that are square-integrable according to Lebesgue

$$\int_a^b dt \, |x(t)|^2 < \infty$$

not necessarily continuous!

L: Lebesgue integral (generalization of Riemann integral)

The elements of $L^2[a,b]$ are Equivalence Classes of functions

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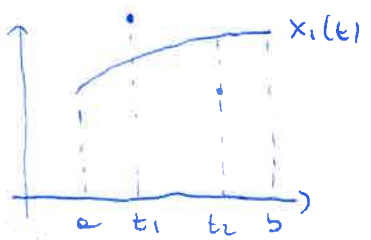
Recall : Equivalence relation in an arbitrary set X : collection of ordered pairs R of X of the kind (a,b) with $a,b \in X$. If $(a,b) \in R$ we say that $a \sim b$. To be called equivalence relation it must fulfill:

- $a \sim a$ (reflexivity)
- $a \sim b \Leftrightarrow b \sim a$ (symmetry)
- if $a \sim b$ and $b \sim c \Rightarrow a \sim c$ (transitivity)

An equivalence relation allows us to split the set into classes.

$X(t) \sim Y(t)$ if $X(t) = y(t)$ almost everywhere (more precisely: except a set with vanishing integral



$x_2(t) = x_1(t) \quad t \neq t_1$

$x_3(t) = x_1(t) \quad t \neq t_2$

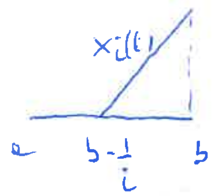
$x_1 \sim x_2 \qquad x_1 \sim x_3$

If $x_4(t) = x_1(t)$ everywhere except $t_a < t < t_b \Rightarrow x_1 \not\sim x_4$

If $x_1(t) \sim 0 \Rightarrow x_1(t) = 0$ almost everywhere $\Rightarrow \int_a^b x_1(t)\, dt = 0$

$L^2[a,b]$ with the scalar product $\langle x|y\rangle = \int_a^b x^*(t)\, y(t)\, dt$ it is complete $\Rightarrow$ HS!

Going back to our previous example



$x_i \to 0$ in $L^2[a,b]$

$L^2[a,b]$ wave functions in quantum mechanics!

DUAL SPACE

$B(X,Y)$ space of all the bounded linear operators from $M_X = (X, \|\cdot\|_X) \to M_Y = (Y, \|\cdot\|_Y)$

it is itself a normed space $T_i \in B(X,Y)$

T_1+T_2 , αT , $\|T\|$

it exists because the operators are bounded

We can consider as a special case the case $Y = \mathbb{R}$ or $\mathbb{C}$ (with the standard metric)

$B(X,\mathbb{K})$ $(\mathbb{K}=\mathbb{R}$ or $\mathbb{C})$ is thus the space of linear continuous (bounded) functionals on X

DEF $M(X, \|\cdot\|)$ normed space $\tilde{M}$ DUAL SPACE :

space of all the continuous linear functionals $\tilde{M} = B(X,\mathbb{K})$

$\Rightarrow$ IT IS A BANACH SPACE (regardless on whether M is BS or not!)

LINEAR OPERATORS ON HILBERT SPACES

notation $|X\rangle \in H$ "Ket" $\langle X| \in \tilde{H}$ "Bra"

scalar product $\langle Y|X\rangle$

$\langle X| \in \tilde{H}$ is defined with this notation because of the RIESZ REPRESENTATION THEOREM :

Every element of $\tilde{H}$ (continuous linear functional $f: H \to \mathbb{K}$) can be uniquely represented by $f(x) = \langle Y|X\rangle$ $Y \in H$

- We start by considering the case $\dim H < \infty$

$\Rightarrow$ linear operators become essentially MATRICES $\Rightarrow$ we can use the standard framework of linear algebra

$\{|\varphi_i\rangle\}_{i=1..N}$ orthonormal basis $\langle \varphi_i|\varphi_j\rangle = \delta_{ij}$

$$\mathbb{1} = \sum_{i=1}^{m} |\varphi_i\rangle\langle\varphi_i| \qquad |x\rangle = \mathbb{1}\,|x\rangle = \sum_{i=1}^{m} |\varphi_i\rangle\langle\varphi_i|x\rangle = \sum_i c_i |\varphi_i\rangle$$

expansion coefficient

$$H \sim \mathbb{K}^m \qquad |x\rangle \sim (c_1 \dots c_m)$$

$$T|x\rangle = T\sum_{j=1}^{m} c_j|\varphi_j\rangle = \sum_{j=1}^{m} c_j\,\underbrace{(T|\varphi_j\rangle)}_{\in H} \qquad T\varphi_j = \sum_i T_{ij}|\varphi_i\rangle$$

$$= \sum_{ij=1}^{m} c_j T_{ij}|\varphi_i\rangle$$

$\Rightarrow$ WE CAN SHOW THAT ALL LINEAR OPERATORS ON H, $\dim H < \infty$ ARE BOUNDED (CONTINOUS)

$$\{\|Tx\|,\ \|x\|=1\} = \left\|\sum_{ij=1}^{m} c_j T_{ij}|\varphi_i\rangle\right\| < \sum_{ij} |c_j|\,|T_{ij}| < \sum_{ij} |T_{ij}|$$

GRAM-SCHMIDT procedure

Given m linearly independent vectors $\{\underline{v}_i\}$ we can construct an ortonormal basis

$$\underline{u}_1 = \underline{v}_1$$

$$\underline{u}_2 = \underline{v}_2 - \frac{\langle \underline{v}_2|\underline{u}_1\rangle}{\|\underline{u}_1\|^2}\,\underline{u}_1 \qquad \underline{e}_i = \frac{\underline{u}_i}{\|\underline{u}_i\|}$$

$$\underline{u}_3 = \underline{v}_3 - \frac{\langle \underline{v}_3|\underline{u}_1\rangle}{\|\underline{u}_1\|^2}\,\underline{u}_1 - \frac{\langle \underline{v}_3|\underline{u}_2\rangle}{\|\underline{u}_2\|}\,\underline{u}_2$$

$\dots$

$u_2 = v_2 - \alpha u_1$

$\langle u_2|u_1\rangle = \langle v_2|u_1\rangle - \alpha\|u_1\|^2 = 0$

$\Rightarrow \alpha = \dfrac{\langle v_2|u_1\rangle}{\|\underline{u}_1\|^2}$

Roughly speaking: subtract from the m vector its projection onto the previous $m-1$ vectors

- EIGENVALUES AND EIGENVECTORS

if $\exists\ |x\rangle \neq 0$ and $T|x\rangle = \lambda|x\rangle$ $\qquad |x\rangle$ eigenvector of T with eigenvalue λ

- RESOLVENT OPERATOR

If $R_T(\lambda) \equiv (T - \lambda I)^{-1}$ exists, it is called RESOLVENT

if $R_T(\lambda)$ exists $\Rightarrow$ λ is not an eigenvalue!

- SYMMETRIC OPERATORS

T linear operator on H is symmetric if $\forall x, y \in D_T$ $\quad \langle Tx|y\rangle = \langle x|Ty\rangle$

- ADJOINT

T bounded operator on H $\Rightarrow$ $\exists\ T^+ \mid \forall x, y \in H$ $\quad \langle y|Tx\rangle = \langle T^+y|x\rangle$

Indeed let us fix $y \in H$ and define $Lx = \langle y|Tx\rangle$

$\Rightarrow$ L is a bounded linear functional on H $\Rightarrow$ according to the Riesz theorem $\exists!\ h \in H \mid Lx = \langle h|x\rangle$

Defining $h = T^+y$ we have $\langle y|Tx\rangle = \langle T^+y|x\rangle$

One can show that T^+ is also linear and bounded

- HERMITIAN OPERATORS

A linear operator T on H is Hermitian (self-adjoint) if $T = T^+$

In the case in which $\underline{\dim H < \infty}$ $\quad$ Symmetric $\Leftrightarrow$ Hermitian

Indeed for linear T we have

$$\langle \varphi_i | T\varphi_j\rangle \equiv T_{ij} = \langle T^+\varphi_i|\varphi_j\rangle = \langle \varphi_j | T^+\varphi_i\rangle^* = (T^+)^*_{ji}$$

If T is symmetric we have

$$\langle \varphi_i | T\varphi_j \rangle = \langle T\varphi_i | \varphi_j \rangle \Rightarrow T_{ij} = T_{ji}^* \Rightarrow T = T^\dagger$$

If $T = T^\dagger$ the eigenvalues of T are <u>real</u>. Indeed suppose $T|u\rangle = \lambda |u\rangle$

$$\Rightarrow \langle u | Tu \rangle = \lambda \|u\|^2 = \langle T^\dagger u | u \rangle = \langle Tu | u \rangle = \langle u | Tu \rangle^* \Rightarrow \lambda = \lambda^*$$

We can choose an orthonormal set $\{|\chi_i\rangle\}$ such that $T|\chi_i\rangle = \lambda_i |\chi_i\rangle$

$$T|x\rangle = T \cdot \mathbb{1} |x\rangle = T \sum_{i=1}^{n} |\chi_i\rangle\langle \chi_i | x\rangle = \sum_i \lambda_i \langle \chi_i | x \rangle |\chi_i\rangle$$

$$= \sum_i \lambda_i P_i |x\rangle$$

$P_i = |\chi_i\rangle\langle \chi_i|$ if λ_i has multiplicity 1

$P_i = \sum_{e=1}^{m_i} |\chi_{ei}\rangle\langle \chi_{ei}|$ if $m_i > 1$

The projectors are <u>idempotent</u>: $P_j^2 = P_j$ and $P_j P_e = 0$ if $j \neq e$

Moreover (still $\dim H < \infty$) we can decompose H as

$$H = H_1 \oplus H_2 \oplus \dots H_M \quad \leftarrow \text{direct sum}$$

<u>HOW TO EXTEND THIS TO ARBITRARY HILBERT SPACES ?</u>

<u>ORTHOGONAL COMPLEMENT</u>

We now move to consider the case in which $\dim H$ is arbitrary

H Hilbert space $\quad Y \subset H$ subspace $\quad (y_1, y_2 \in Y \quad \alpha y_1 + \beta y_2 \in Y)$

If Y is <u>closed</u> $\Rightarrow$ Y is <u>complete</u> $\Rightarrow$ Y is also a Hilbert space

Indeed let us consider $x_n \in Y$ of Cauchy type $\Rightarrow$ $x_n \to x \in H$ since H is complete. But since Y is <u>closed</u> then $x \in Y$

EXAMPLE

$H = L^2[a,b)$ $\quad Y = C[a,b]$ $\quad Y \subset H$ and subspace but it is not complete!

$\Rightarrow$ it is not a Hilbert space!

Consider a closed subspace $Y \subset H$ $\Rightarrow$ the orthogonal complement $Y^\perp$ is defined as $Y^\perp \equiv \{ z \in H \mid z \perp y \ \forall y \in Y \}$

Note that $Y \cap Y^\perp = \{ |0\rangle \}$

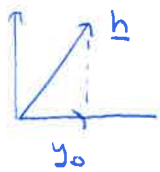
DIRECT SUM

We say that $H = Y \oplus Z$ if $\forall x \in H \ \exists! \, y \in Y$ and $z \in Z \mid x = y + z$

PROJECTION THEOREM

H Hilbert space, Y closed subspace $\Rightarrow \forall h \in H \ \exists! \, y_0 \in Y \mid h - y_0 \perp z \ \forall z \in Y$

y_0 is said to be the projection of h on Y



PROOF

We start by showing that if y_0 exists, then it is unique.

Let y_1 and y_2 two projections of h $\Rightarrow \forall y \in Y$ $\quad \langle h - y_1 | y \rangle = 0$

and $\langle h - y_2 | y \rangle = 0$ $\Rightarrow \langle y_1 - y_2 | y \rangle = 0$ but since y is arbitrary, set $y = y_1 - y_2$

$\Rightarrow \| y_1 - y_2 \| = 0 \Rightarrow y_1 = y_2$.

We now have to prove the existence of the projection. Consider

$d = \inf \{ \| h - y \|, y \in Y \}$ and define a sequence $y_m \in Y$ such that

$$d \leq \| h - y_m \| < d + \frac{1}{m}$$

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By using the parallelogram rule we can show that y_n is a Cauchy sequence

$$\|y_n - y_m\|^2 = \|\overset{X}{(y_n-h)} + \overset{Y}{(h-y_m)}\|^2 = 2\left(\|y_n-h\|^2 + \|h-y_m\|^2\right)$$

$$-\|\overset{X}{(y_n-h)} - \overset{Y}{(h-y_m)}\|^2 = 2\left(\|y_n-h\|^2 + \|h-y_m\|^2\right) - 4\underbrace{\left\|\frac{y_n+y_m}{2} - h\right\|^2}_{\geq d^2}$$

Now given $\varepsilon > 0$ $\exists N(\varepsilon)$ | $\|y_n - h\|^2 < d^2 + \frac{\varepsilon}{4}$ if $n > N(\varepsilon)$

and $\|y_m - h\|^2 < d^2 + \frac{\varepsilon}{4}$ if $m > N(\varepsilon)$

$\Rightarrow$ $\|y_n - y_m\|^2 \leq 2\left(2d^2 + \frac{\varepsilon}{2}\right) - 4d^2 = \varepsilon$ $\Rightarrow$ y_n is of Cauchy type.

Since the Hilbert space is complete y_n converges : let $y_0 = \lim_{n\to\infty} y_n$

We have $d = \lim_{n\to\infty} \|h - y_n\| = \|h - y_0\|$ $\Rightarrow$ $\|h-y_0\| \leq \|h-y\|$ $\forall y \in Y$

This is enough to show that y_0 is the projection. Indeed

$$\|h - y_0 + ty\|^2 = \|h-y_0\|^2 + t^2\|y\|^2 + 2t \,\mathrm{Re}\langle h-y_0|y\rangle$$

y arbitrary
t real

But $\|h-y_0+ty\|^2 - \|h-y_0\|^2$ must always be positive and minimum for $t=0$

Since $\|h-y_0+ty\|^2 - \|h-y_0\|^2$ is a polynomial in t, the linear term has to vanish $\Rightarrow$ $\mathrm{Re}\langle h-y_0|y\rangle = 0$

But y was arbitrary, so we could choose $y \to iy$ $\Rightarrow$ $\langle h-y_0|y\rangle = 0$

$\Rightarrow$ We conclude that y_0 is the projection of h on Y and it is <u>UNIQUE</u> ∎

Given $h \in H$ and $Y \subset H$ closed subspace, we can always write $h = y + z$ with $y \in Y$ and $z \in Y^\perp$

ORTHONORMAL BASIS

When $\dim H = n$ we can find an orthonormal basis by looking for a set of n linearly independent vectors and applying the Gram-Schmidt procedure

How can we do this when $\dim H = \infty$?

$\{|\varphi_i\rangle\}$ orthonormal system

$\text{span}\{|\varphi_i\rangle\} \equiv \{x \mid x = \sum_{i=1}^{n} \alpha_i \varphi_i\}_{n \in \mathbb{N}}$

DEF $\{|\varphi_i\rangle\}$ is COMPLETE if span$\{\varphi_i\}$ is dense in H

It means that every element of H can be approximated as well as we like with an element of $\text{span}\{|\varphi_i\rangle\}$

(alternatively: if $y \in \text{span}\{|\varphi_i\rangle\}^{\perp} \Rightarrow y = 0$)

PROJECTORS

$P : H \to Y \quad | \quad Px = y$

y: projection

H Hilbert space

Y closed subspace

$P^2 = P$ and $\|P\| = 1$

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Separable HS

A Hilbert space is said to be separable if it has a complete numerable orthonormal basis

$\{|\varphi_i\rangle\}_{i=1,2\dots}$

Relevant HS are all separable : they are also the natural extension of HS of finite dimension !

$$|Y\rangle = \sum_{i=1}^{\infty} y_i |\varphi_i\rangle$$

(more precisely it means $|Y_N\rangle = \sum_{i=1}^{N} y_i |\varphi_i\rangle$ $\lim_{N\to\infty} |Y_N\rangle = |Y\rangle \iff \lim_{N\to\infty} \| Y_N - Y \| = 0$

example : $L^2[-1,1]$

$$|M_i\rangle = t^i \qquad -1 \le t \le 1$$

span $\{|M_i\rangle\}$ is the set of polynomials on $[-1,1]$

span $\{|M_i\rangle\}$ is dense in $C[-1,1]$, but $C[-1,1]$ is dense in $L^2[-1,1]$

Using the Gram-Schmidt construction we can construct the LEGENDRE POLYNOMIALS

$P_0(x) = 1$ $\quad P_1(x) = x$ $\quad P_2(x) = \frac{1}{2}(3x^2 - 1)$ $\quad \dots$ $\quad P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m}(x^2-1)^m$

example $L^2(-\infty,\infty)$ $\quad |M_0\rangle = e^{-t^2/2}$ $\quad |M_i\rangle = t\,|M_{i-1}\rangle$

Gram-Schmidt $\to$ Hermite Polynomials

$H_0(x) = 1$
$H_1(x) = 2x$
$H_2(x) = 4x^2 - 2$
$\dots$

example $L^2[0,\infty]$

$|M_0\rangle = e^{-t/2}$ $\quad |M_i\rangle = t\,|M_{i-1}\rangle$

Gram-Schmidt $\to$ Laguerre

- ℓ^2 $\quad x \in \ell^2$ $\quad x = (x_1, x_2, \dots x_n \dots)$ $\quad \sum |x_i|^2 < \infty$ $\quad x_i \in \mathbb{C}$

$|\varphi_i\rangle \sim (0 \dots 1, 0 \dots)$ complete orthonormal system
$\uparrow$ i position

SEPARABLE

All dim $= \infty$ HS are ISOMORPHIC TO ℓ^2

example $\quad H = L^2[-\pi,\pi] \qquad |\varphi_k\rangle = \frac{1}{\sqrt{2\pi}} e^{ikt} \quad , k \in \mathbb{Z}$ is an orthonormal set

We know that any continuous function in $[-\pi,\pi]$ with continuous derivative can be expanded in Fourier series. But these functions form a dense set in $L^2[-\pi,\pi]$

$\Rightarrow \quad \{|\varphi_n\rangle\}$ is a complete basis

example: non separable space

$$t \to e^{ist} \qquad t \in \mathbb{R} \quad s \text{ real parameter}$$

We consider the linear space of these functions with the product $\langle f, g\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f^*(t)\, g(t)$

we have that taken $f(t) = e^{ist} \quad g(t) = e^{irt} \quad s \neq r$

$$\langle f|g\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{i(s-r)t}\, dt = 0 \qquad \|g\| = 1$$

$\Rightarrow$ These functions provide a non numerable basis for the HS $\Rightarrow$ NON SEPARABLE

- CLOSED, BOUNDED, COMPACT OPERATORS

Up to now we have considered, in the class of linear operators, those that are BOUNDED.

We have seen that if dim H is finite $\Rightarrow$ all linear operators are indeed bounded.

However in Physics not all important operators are BOUNDED!

example $\quad \psi(x) \in L^2[0,1] \qquad |\psi_m\rangle \sim \psi_m(x) = x^m \qquad \|\psi_m\|^2 = \int_0^1 x^{2m} dx = \frac{1}{2m+1}$

$P = -i\hbar \frac{\partial}{\partial x}$ momentum operator *

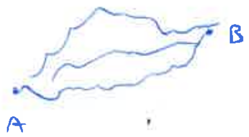
$$\|\psi_m\| \to 0 \qquad P|\psi_m\rangle \sim m x^{m-1}$$

$\Rightarrow \quad \|P\psi_m\| = \frac{m}{\sqrt{2m-1}}$ and there is no constant C $\mid \quad \|P\psi_m\| \leq C\|\psi_m\|$!

However the important operators in physics are CLOSED

2) WAVE OPTICS | WAVE MECHANICS

light = em field

ψ  $\sum_{\text{paths}} e^{i\frac{S}{\hbar}}$

$\lambda \to 0$ | $S \gg \hbar$

GEOMETRIC OPTICS

light = rays

← classical trajectory

$\hat{p} = \alpha \frac{\partial}{\partial x}$ $\qquad \psi_{cl} = e^{i\frac{S}{\hbar}}$ $\qquad \hat{p}\psi_{cl} = \alpha \frac{i}{\hbar}\frac{\partial S}{\partial x}\psi_{cl}$

⇒ but classical momentum is $\underline{p} = \nabla S^{*}$ ⇒ $\alpha = -i\hbar$

1) Translation operator $\delta\psi = \frac{\partial \psi}{\partial \underline{r}}\delta \underline{r}$ $\qquad T \sim \underline{\nabla}$

Invariance under translation ⇒ conservation of momentum ⇒ $\underline{p}$ must be $\sim \underline{\nabla}$

(*) $$\delta S = \frac{\partial L}{\partial \dot{x}}\delta x\Big|_{t_1}^{t_2} + \int_{t_1}^{t}\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)\delta x \, dt$$

⇒ consider physical trajectories that fulfill the Euler equations

but such that $\delta x(t_1) = 0$ and $\delta x(t_2) = \delta x$ is arbitrary

⇒ $$\frac{\partial S}{\partial x} = p = \frac{\partial L}{\partial \dot{x}}$$

CLOSED OPERATORS

DEF Let us consider an operator $T : D_T \subset H \to H$ linear. T is closed if $\forall x_n \in D_T$ with $x_n \to x \in H$ and $Tx_n \to y$ we have $x \in D_T$ and $y = Tx$

Note that BOUNDED $\not\Rightarrow$ CLOSED

EXAMPLE

$H = \mathbb{R}$, $D_T = \mathbb{Q} \subset \mathbb{R}$ $\quad Tx = x \quad$ T is bounded $(\|T\| = 1)$ but not closed

$x_n = \left(1 + \frac{1}{n}\right)^n \in D_T \quad x_n \to x \,(= e) \quad Tx_n \to y \,(= e)$ but $x \notin D_T$

A linear bounded operator is closed if D_T is a closed set.

COMPACT OPERATORS

We know that if X is a normed space and $\dim X < \infty \Rightarrow$ the Bolzano-Weierstrass theorem holds

$\forall x_n \in X$, x_n bounded $\exists$ a x_{n_k} subsequence of x_n which is convergent

However this does not hold if $\dim X = \infty$!

EXAMPLE

$X = \mathbb{R}^n \qquad S^n = \{ x \in X \mid \sum_{i=1}^{n} x_i^2 = 1 \}$ sphere in $\mathbb{R}^n$

given $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, \dots x_n^{(i)})$ sequence with $i = 1, 2 \cdots \infty$

$\exists$ convergent subsequence (points have to accumulate somewhere!)

take now $n = \infty$ and consider

$$x^{(i)} = (0, 0 \,\dots\, \underset{\substack{\uparrow \\ i \text{ position}}}{1}, 0 \dots 0 \dots) \qquad x^{(i)} \in S^\infty \qquad \text{bounded sequence}$$

$\nexists$ a convergent subsequence! (points "spread" across the infinite dimensions)

This result motivates us to give the following definition:

DEF X normed linear space, $A \subset X$ is **RELATIVELY COMPACT**

if $\forall x_n \in A$ $\exists$ x_{n_k} subsequence which is **CONVERGENT**

A relatively compact set A is bounded. In fact, suppose it is not. Then there must be a sequence $x_n \in A$ | $\lim_{n\to\infty} \|x_n\| = \infty$. But such a sequence would not have a convergent subsequence.

A subset $A \subset X$ is **COMPACT** if it is **RELATIVELY COMPACT** and **CLOSED**

We can now define a **COMPACT OPERATOR**

T BOUNDED : bounded set $M \xrightarrow{T}$ bounded set M'

DEF T COMPACT : bounded set $M \xrightarrow{T}$ relatively compact set M'

A compact operator is bounded, since a relatively compact set is bounded

A bounded operator is not necessarily compact (e.g. the identity operator is bounded but not compact)

$$\boxed{\text{COMPACT} \;\Rightarrow\; \text{BOUNDED} \qquad \not\Leftarrow}$$

eg: momentum operator is not bounded $\Rightarrow$ not compact!

EXAMPLE

$T_N : \ell^2 \to \ell^2$ $\quad x = (x_1, x_2 \dots)$ $\quad T_N x = (x_1, x_2 \dots x_N, 0, 0 \dots)$

T_N is compact. Indeed its range is $\mathbb{C}^N$ and it is finite dimensional

$\Rightarrow$ a bounded set is sent into a relatively compact set for the Bolzano-Weierstrass theorem.

EXAMPLE

The identity operator in $\ell^2 \to \ell^2$ is not compact.

Consider $x_n = (0, 0 \dots 1, 0, 0 \dots)$ $\quad x_n$ is bounded
$\uparrow$ n position

But there exists no subsequence of $I x_n$ which is convergent.
(indeed for $m \neq n$ $\quad \| I x_n - I x_m \| = \sqrt{2} \Rightarrow$ any subsequence of $I x_n$ cannot be a Cauchy sequence and thus does not converge)

HELLINGER-TOEPLITZ THEOREM

A linear operator on H is said to be symmetric if $\forall x, y \in D_T$ $\quad \langle y | T x \rangle = \langle T y | x \rangle$

Let us suppose that $D_T = H \Rightarrow$ one can prove that T is BOUNDED

SYMMETRIC vs HERMITIAN OPERATORS

We have seen that for $\dim H < \infty$ $\quad$ SYMMETRIC $\Leftrightarrow$ HERMITIAN

$\Rightarrow$ let us see that this is not the case when $\dim H = \infty$

EXAMPLE

$$H = L^2[0,1] \qquad A = i \frac{d}{dt} \qquad D_A = \{ x \in L^2[0,1] \mid \exists\, x' \in L^2[0,1] \}$$

$$A_0 = i \frac{d}{dt} \qquad D_{A_0} = D_A \cap \{ x \in L^2(0,1) \mid x(0) = 0 , x(1) = 0 \}$$

$$\langle y | A_0 x \rangle = \int_0^1 dt\, y^*(t)\, i \frac{d}{dt} x(t) = \underbrace{i\, y^*(t)\, x(t) \Big|_0^1}_{=0} + \int_0^1 dt \left(i \frac{d}{dt} y(t) \right)^* x(t)$$

$$= \langle A_0 y | x \rangle$$

$\Rightarrow$ A_0 is symmetric $\qquad$ but $A_0^+ = A \; (\neq A_0)$! $\qquad$ so A_0 is not hermitian!

Indeed

$$\langle y | A_0 x \rangle = \underbrace{i\, y^*(t)\, x(t) \Big|_0^1}_{\text{does not need to vanish}} + \int_0^1 dt \left(i \frac{d}{dt} y(t) \right)^* x(t) = \langle A y | x \rangle$$

More generally we can say that if A is symmetric $\Rightarrow$ $A \subset A^{+}$ that is

$A^{+}|x\rangle = A|x\rangle \quad \forall x \in D_A$ but $D_{A^+} \supset D_A$

$$\boxed{\text{HERMITIAN} \Rightarrow \text{SYMMETRIC} \\ \not\Leftarrow}$$

FURTHER PROPERTIES

- We have seen that if a linear operator T is bounded $\Rightarrow \exists!\, T^{+}$

  One can prove that T^{+} is <u>CLOSED</u>

- T_1 bounded, T_2 compact $\Rightarrow$ <u>$T_2 \cdot T_1$ is compact</u>

  Indeed:

  $$M \text{ bounded} \xrightarrow{T_1} M_1 \text{ bounded} \xrightarrow{T_2} M_2 \text{ relatively compact}$$

  $$\Rightarrow T_2 T_1 \text{ is compact!}$$

SPECTRUM

T linear operator Resolvent is $R_T(\lambda) = (T - \lambda I)^{-1}$ (if it exists!)

$\lambda \in \mathbb{C}$ (or $\mathbb{R}$) is a regular point if $R_T(\lambda)$ exists, and is defined on all H and bounded

$\rho(T) = \{\lambda \mid \lambda$ is a regular point$\}$ Resolvent set

$\sigma(T) = \mathbb{K} - \rho(T)$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$) is the Spectrum of T

if λ is an eigenvalue of T $\Rightarrow$ $R_T(\lambda)$ does not exist $\Rightarrow$ $\lambda \in \sigma(T)$

the set of eigenvalues of T is called $\sigma_d(T)$ discrete spectrum

if λ is such that $R_T(\lambda)$ exists, but it is not bounded, or not defined over all H

$\Rightarrow$ $\lambda \in \sigma_c(T)$ continuous spectrum

$\sigma(T) = \sigma_d(T) + \sigma_c(T)$

example $H = \mathbb{C}^n$ $\sigma(T) = \sigma_d(T) = \{\lambda_i\}$ λ_i eigenvalues

$\sigma_c(T) = \emptyset$

if λ is not an eigenvalue $\Rightarrow$ $(T - \lambda I)^{-1}$ exists and its domain is $H = \mathbb{C}^n$

example $H = \ell^2$ $T_\rightarrow : (x_1, x_2 \cdots) \to (0, x_1, x_2 \cdots)$

$T|x\rangle = \lambda|x\rangle$ $0 = \lambda x_1$ $x_1 = \lambda x_2$ ... $\Rightarrow x_1 = 0,\ x_2 = 0 \ldots$

$\Rightarrow T_\rightarrow$ has no eigenvalues!

$(T-\lambda)|x\rangle = (-\lambda x_1, x_1 - \lambda x_2, x_2 - \lambda x_3, \cdots)$ $(T-\lambda)^{-1}(-\lambda x_1, x_1 - \lambda x_2 \cdots) = (x_1, x_2 \cdots)$

or $(T-\lambda)^{-1}(y_1, y_2 \cdots y_n \cdots) = \left(-\frac{y_1}{\lambda}, -\frac{y_2}{\lambda} + \frac{y_1}{\lambda^2}, \cdots\right)$

$\Rightarrow$ T^{-1} does not exist and $R_T(0)$ does not exist

$\Rightarrow$ $\lambda = 0 \in \sigma(T)$ actually $\sigma(T) = \{\lambda : |\lambda| \leq 1\}$

SPECTRUM OF COMPACT OPERATORS

We have seen that if T is a linear compact operator on H $\Rightarrow$ it is bounded.

$\Rightarrow$ $\|Tx\| \leq \|T\| \|x\|$ $\Rightarrow$ if λ is an eigenvalue $\|Tx\| = |\lambda| \|x\|$ $\Rightarrow$ $|\lambda| \leq \|T\|$

We conclude that either there is a finite number of eigenvalues, or there must be an accumulation point. We now want to show that $0 \in \sigma(T)$ if $\dim H = \infty$.

Indeed suppose that $\lambda = 0$ is a regular point for T

$\Rightarrow$ T^{-1} exists, it is bounded and defined over all H.

But the product of a bounded and a compact operator is compact

$\Rightarrow$ $I = T^{-1}T$ is compact $\Rightarrow$ BUT THIS IS NOT POSSIBLE IF $\dim H = \infty$!
(see example in ℓ^2)

$\Rightarrow$ we must have $0 \in \sigma(T)$

We can formulate the following Theorem

— SPECTRUM OF COMPACT OPERATORS

T compact operator on H, such that $\dim H = \infty$

- $0 \in \sigma(T)$
- Each $\lambda \in \sigma(T)$, $\lambda \neq 0$ is an eigenvalue of T with finite dimensional eigenspace
- $\sigma(T)$ is either a finite set or a sequence converging to 0

EXAMPLE

$H = L^2[0,1]$ $\quad (Tx)(t) = \int_0^t x(s)\,ds$ $\quad$ One can show that T is compact

T^{-1} is the derivative operator $\Rightarrow$ not bounded! $\Rightarrow$ $0 \in \sigma(T)$

$\forall \lambda \neq 0$ $\quad (T - \lambda I)^{-1}$ exists and is bounded (solve the equation $(T-\lambda I)x = y$)

$\Rightarrow$ $\sigma(T) = \{0\}$ in this case the spectrum is finite!

SPECTRUM OF COMPACT HERMITIAN OPERATORS

Let T be a linear operator on H with $T=T^\dagger$ and T compact. This is the case in which the simple extension of what happens for $H=\mathbb{C}^n$ holds.

- We start by showing that $\|T\| = \sup_{\|x\|=1} |\langle x|Tx\rangle| \equiv \sigma$. We first prove that $\|T\| \geq \sigma$. We have $|\langle x|Tx\rangle| \leq \|x\|\,\|Tx\| \leq \|x\|\,\|T\|\,\|x\| \Rightarrow \sigma \leq \|T\|$

To show the opposite we define ($k\in\mathbb{R}$)

$$|v_+\rangle = k|x\rangle + \frac{1}{k}T|x\rangle$$

$$|v_-\rangle = k|x\rangle - \frac{1}{k}T|x\rangle$$

We have $\|Tx\|^2 = \frac{1}{4}\left(\langle Tv_+|v_+\rangle - \langle Tv_-|v_-\rangle\right) \leq \frac{1}{4}\left(|\langle Tv_+|v_+\rangle| + |\langle Tv_-|v_-\rangle|\right)$

$$\leq \frac{\sigma}{4}\left(\|v_+\|^2 + \|v_-\|^2\right) = \frac{\sigma}{2}\left(k^2\|x\|^2 + \frac{1}{k^2}\|Tx\|^2\right)$$

By taking the derivative with respect to k^2 we find that the last expression is minimum when $k^2 = \frac{\|Tx\|}{\|x\|}$ and we get

$$\|Tx\|^2 \leq \frac{\sigma}{2}\,2\,\|Tx\|\,\|x\| \quad \text{that is} \quad \|Tx\| \leq \sigma\|x\| \Rightarrow \|T\| \leq \sigma$$

- We now want to show that the eigenvalue with largest modulus is $|\lambda_1| = \|T\|$

Let x_n with $\|x_n\|=1$, $x_n\in H$ | $|\langle x_n|Tx_n\rangle| \to \|T\|$ (x_n exists for the definition of sup!)

Since $T=T^\dagger$ $\langle x_n|Tx_n\rangle$ is real $\Rightarrow$ this implies $\langle x_n|Tx_n\rangle \to \underbrace{\pm\|T\|}_{\lambda_1}$

We have $\|Tx_n - \lambda_1 x_n\|^2 = \langle Tx_n-\lambda_1 x_n, Tx_n - \lambda_1 x_n\rangle$

$$= \|Tx_n\|^2 - \lambda_1^* \langle x_n|Tx_n\rangle - \lambda_1\langle x_n|Tx_n\rangle + |\lambda_1|^2 \overset{1}{\|x_n\|^2}$$

But since both $\langle Tx_n|x_n\rangle$ and λ_1 are real $\lambda_1\langle x_n|Tx_n\rangle = |\lambda_1\langle x_n|Tx_n\rangle|$

$$= \|Tx_n\|^2 - 2|\lambda_1\langle x_n|Tx_n\rangle| + |\lambda_1|^2 \leq \|T\|^2 - 2|\lambda_1|\,|\langle x_n|Tx_n\rangle| + |\lambda_1|^2$$

$$= 2|\lambda_1|\left(|\lambda_1| - |\langle x_n|Tx_n\rangle|\right) \to 0$$

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Since x_m is bounded and T is compact $\Rightarrow \exists$ a convergent subsequence $Tx_{m_k} \Rightarrow \lambda_1$ is eigenvalue of T. Having shown that there exists an eigenvalue λ_1 with $|\lambda_1| = \|T\|$, we can consider $H_1 = \text{span}\{x_i\}$ where x_i are the eigenvectors of λ_1, and construct $H_1^\perp$. We can then focus on $T' C T$ where T' is the restriction of T to $H_1^\perp$. We have that T' is also compact and $\|T'\| \leq \|T\|$

$\Rightarrow$ we can repeat the construction done for T and find λ_2 with $|\lambda_2| \leq |\lambda_1|$

$\Rightarrow$ we can write $\quad H = \underbrace{H_1 \oplus H_2 + \cdots H_n}_{\dim H_i < \infty} \oplus \cdots H_0$

H_0: eigenspace corresponding to $\lambda = 0$ ($\dim H_0 = \infty$ is possible)

$$\Rightarrow \boxed{T = \sum_{i=1}^{\infty} \lambda_i |\varphi_i\rangle\langle\varphi_i| + (0 \cdot P_0)}$$

SPECTRAL REPRESENTATION

$\lambda_i \in \mathbb{R} \qquad |\lambda_i| \to 0 \qquad \sigma(T) = \sigma_d(T) = \{\lambda_i\} \cup \{0\}$
$\cup\{0\}$

EXAMPLE (*)

$H = \ell^2 \qquad x = (x_1, x_2 \ldots) \qquad T|x\rangle = (x_1, \frac{x_2}{2}, \frac{x_3}{3} \ldots) \qquad T = T^+ \quad D_T = \ell^2$

$\sum_i |x_i|^2 < \infty \qquad T$ compact

$\lambda_i = \frac{1}{i} \quad i \in \mathbb{N}$ corresponds to the eigenvector $|\varphi_i\rangle = (0, 0 \ldots 1, 0 \ldots)$

$\uparrow$ i position

$T|\varphi_i\rangle = \frac{1}{i}|\varphi_i\rangle$

$T = \sum_{i=1}^{\infty} \lambda_i P_i = \sum_{i=1}^{\infty} \frac{1}{i} P_i \qquad P_i = |\varphi_i\rangle\langle\varphi_i| \qquad H = \bigoplus_{i=1}^{\infty} H_i$

$\lambda_i \to 0 \qquad 0$ is an accumulation point! $\qquad \dim H_i = 1$

(x) in this case we have $\sigma(T) = \{ \frac{1}{i} , i \in \mathbb{N} \} \cup \{0\}$ $\sigma_c = \{0\}$

$\lambda = 0$ is NOT an eigenvalue but T^{-1} is not bounded!

and it does not exist over the whole H

(for example if $x_n = \frac{1}{n}$ $\sum |x_n|^2 < \infty$, so $x_n \in \ell^2$

but $T^{-1} x_n = (1, 1, 1 \cdots) \notin \ell^2$)

THE RESOLVENT

T linear operator on H. If $R_T(\lambda)$ exists, bounded, and $D_{R_T(\lambda)} = H$ $\Rightarrow$ λ is a regular point

$$(T-\lambda I)\, R_T(\lambda) = I_H \qquad\qquad R_T(\lambda)\,(T - I\lambda) = I_{D_T}$$

$$\Rightarrow \quad R_T(\lambda)\, T \subseteq T R_T(\lambda) = (T - \lambda I + \lambda I) R_T(\lambda) = I + \lambda R_T(\lambda)$$

Resolvent identity

$R_T(\lambda_1) \equiv R_1 \qquad R_T(\lambda_0) \equiv R_0$ Let's start from the identity $(T-\lambda_0 I) - (T - \lambda_1 I) = (\lambda_1 - \lambda_0) I$

- multiply by R_0 on the right $\rightarrow$ $I - (T - \lambda_1 I) R_0 = (\lambda_1 - \lambda_0) R_0$
- multiply by R_1 on the left $\rightarrow$ $\boxed{R_1 - R_0 = (\lambda_1 - \lambda_0) R_1 R_0}$

we can use this equation to find an iterative solution for the resolvent

$R_1 = R_0 + (\lambda_1 - \lambda_0) R_1 R_0$ but if we swap 0 and 1 we get $R_0 = R_1 + (\lambda_0 - \lambda_1) R_0 R_1$

$\Rightarrow R_1 = R_0 + (\lambda_1 - \lambda_0) R_0 R_1 \quad \Rightarrow \quad [R_T(\lambda_1), R_T(\lambda_2)] = 0$ if $\lambda_1, \lambda_2 \in \rho(T)$

$$R_1 = R_0 + (\lambda_1 - \lambda_0) R_0 R_1 = R_0 + (\lambda_1 - \lambda_0) R_0 \left(R_0 + (\lambda_1 - \lambda_0) R_0 R_1 \right)$$
$$= R_0 + (\lambda_1 - \lambda_0) R_0^2 + (\lambda_1 - \lambda_0)^2 R_0^2 R_1 + \dots$$

$$\Rightarrow \quad R_T(\lambda) = \sum_{m=0}^{\infty} (\lambda - \lambda_0)^m \left(R_T(\lambda_0) \right)^{m+1} \qquad \lambda, \lambda_0 \in \rho(T)$$

Perturbation theory

$H = H_0 + V$ — H_0 "easy to solve" — $H\psi = \lambda\psi$

V perturbation (small!) — $H_0 \psi = \lambda_0 \psi$

$$H - \lambda I = H_0 - \lambda I + V$$

$$I = R_H(\lambda) \left[H_0 - \lambda I + V \right]$$

$$R_{H_0}(\lambda) = R_H(\lambda) + R_H(\lambda)\, V\, R_{H_0}(\lambda)$$

$$R_H(\lambda) = R_{H_0}(\lambda) - R_H(\lambda)\, V\, R_{H_0}(\lambda)$$

$$\Rightarrow \quad R_H(\lambda) = R_{H_0}(\lambda) \sum_{m=0}^{\infty} \left(-V R_{H_0}(\lambda)\right)^m$$

Perturbative expansion

OPERATORS WITH COMPACT RESOLVENT

Suppose that T is a linear operator on H and that $R_0 \equiv R_T(\lambda_0) = (T - \lambda_0 I)^{-1}$ is compact for $\lambda_0 \in \rho(T)$. This implies that R_0 is bounded and defined over all $H \Rightarrow$ We can show that $\forall \lambda \in \rho(T)$ $R_T(\lambda)$ is also compact

Use the resolvent identity: $R_T(\lambda) - R_T(\lambda_0) = (\lambda - \lambda_0) R_T(\lambda) R_T(\lambda_0)$

$$R_T(\lambda) = \underbrace{\left[1 + (\lambda - \lambda_0) R_T(\lambda)\right]}_{\text{bounded}} \underbrace{R_T(\lambda_0)}_{\text{compact}} \quad \Rightarrow \quad \text{the product is compact}$$

From the spectral theorem for compact operators we can say that $R_T(\lambda)$ has a discrete spectrum, with possibly 0 as accumulation point $\sigma(R_T(\lambda)) = \sigma_d(R_T(\lambda))$

CAN WE SAY SOMETHING ON THE SPECTRUM OF T ?

Let us consider $Q \equiv -\lambda R_0 (R_0 - \lambda I)^{-1}$ bounded for $\lambda \in \rho(R_0)$ and defined over all H

$$Q(R_0 - \lambda I) = -\lambda R_0 \quad \Rightarrow \quad \lambda(Q - R_0) = Q R_0 \quad \Rightarrow \quad Q - R_0 = \frac{1}{\lambda} Q R_0 \qquad \lambda \neq 0$$

But this is the equation fulfilled by $R_T\left(\lambda_0 + \frac{1}{\lambda}\right)$! (*)

$$\Rightarrow \quad Q = -\lambda R_0 (R_0 - \lambda I)^{-1} = R_T\left(\lambda_0 + \frac{1}{\lambda}\right)$$

bounded and defined all over H if $\lambda \neq 0$ $\lambda \in \rho(R_0)$

$$\Rightarrow \quad \boxed{\lambda_0 + \frac{1}{\lambda} \in \rho(T) \iff \lambda \in \rho(R_0), \; \lambda \neq 0}$$

$\Rightarrow$ $\sigma(T)$ includes $\underbrace{\lambda_0 + \frac{1}{\lambda_m}}$ and possibly the accumulation point at ∞ (since $\lambda_m \to 0$)

λ_m eigenvalues of $R_0 \equiv R_T(\lambda_0)$

EXAMPLE : STURM-LIOUVILLE OPERATOR

Differential operator $L = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$ defined over $L^2[\dots]$

$L = L^+$, L not bounded , but has compact resolvent

$\Rightarrow$ distinct eigenvalues with ∞ as accumulation point

$L|x\rangle = \lambda|x\rangle$ differential equation $\Rightarrow$ prototype of familiar differential equations !

example

$H = L^2[-1,1]$ $p(x) = 1-x^2$ $q(x) = 0$ $Ly = \lambda y$

$\lambda = \ell(\ell+1)$ $\Rightarrow$ Legendre equation $(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$

$\Rightarrow$ eigenfunctions of L are in this case the familiar Legendre Polynomials !

(basis for $L^2[-1,1]$)

$\ell=0 \to 1$
$\ell=1 \to x$
$\ell=2 \to 3x^2-1$
$\dots$

$p = e^{-x^2}$ $q = 0$ weight e^{x^2}

$\uparrow$

Analogously we can treat the case of Laguerre and Hermite polynomials .

SPECTRAL FAMILY AND RESOLUTION OF THE IDENTITY

Goal : extend the spectral representation to hermitian but not necessarily compact operators

Abstract definition

A family of orthogonal projections $E(\lambda)$, $-\infty < \lambda < \infty$ in a Hilbert space H is called a RESOLUTION OF THE IDENTITY (a spectral family) if it satisfies the following conditions

- $E(\lambda)\,E(\mu) = E(\min(\lambda,\mu))$
- $E(-\infty) = 0$ $E(+\infty) = I$
- $\lim_{\varepsilon \to 0^+} E(\lambda+\varepsilon) = E(\lambda)$

27 Let us go back to the case in which <u>T is compact</u>

$$\Rightarrow \quad \sigma(T) = \sigma_d(T) \cup \{0\} \quad , \quad \sigma_c(T) = \emptyset \text{ or } \{0\} \qquad T = \sum_i \lambda_i P_i$$

P_i projectors on the eigenspace corresponding to the eigenvalue λ_i

Let us define $E(\lambda) \equiv \sum_{\lambda_i \leq \lambda} P_i$ operator valued function of λ

More explicitly, suppose $m_1 = \min(\lambda_i)$, $m_2 = \max(\lambda_i)$

$$\Rightarrow \quad E(\lambda) = 0 \quad \lambda < m_1 \; ; \qquad E(\lambda) = \mathbb{I} \quad \lambda > m_2$$

moreover $E^2(\lambda) = E(\lambda)$ and $E^\dagger(\lambda) = E(\lambda)$

we also have for $\lambda \leq \mu$ $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\lambda)$

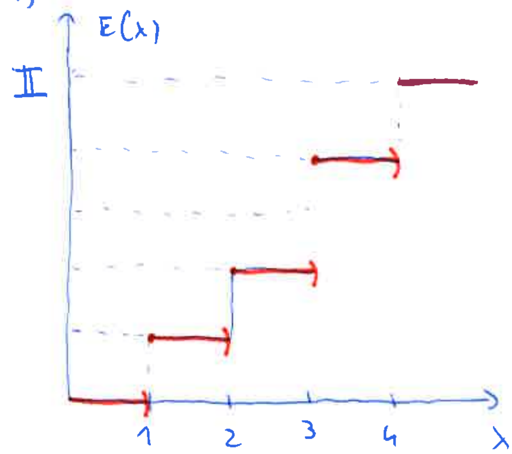
<u>example</u>

$\lambda_1 = 1 \quad \lambda_2 = 2$

$\lambda_3 = 3 \quad \lambda_4 = 4$

$$T = \sum_{i=1}^{4} \lambda_i P_i \qquad m(\lambda_3) = 2$$

multiplicity



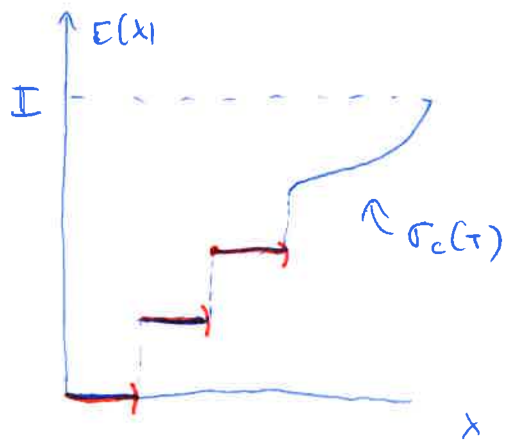
Now consider the case in which the λ_i become closer and closer $\Delta E_i = E(\lambda_i) - E(\lambda_{i-1}) = P_i$

$$T = \sum_i \lambda_i P_i = \sum_i \lambda_i \Delta E_i \rightarrow \int_{-\infty}^{+\infty} \lambda \, dE(\lambda)$$

the sum becomes an integral

In the case of a generic (not necessarily compact or with compact resolvent) hermitian operator T , we have $\sigma_c(T) \neq \emptyset$ $\sigma(T) \subset \mathbb{R}$

$\Rightarrow$ the function $E(\lambda)$ is not anymore a step function!



SPECTRAL THEOREM FOR GENERIC HERMITIAN LINEAR OPERATORS :

$\Rightarrow$ T hermitian operator on H $\Rightarrow$ $\exists\ E(\lambda)$ spectral family $\Big|\ T = \int_{\mathbb{R}} \lambda\, dE(\lambda)$

The real axis is divided as follows :

a) $\lambda_0 \in \rho(T)$ if $E(\lambda)$ is constant around λ_0

b) $\lambda_0 \in \sigma_d(T)$ if $E(\lambda)$ has a step in λ_0

c) $\lambda_0 \in \sigma_c(T)$ if $E(\lambda)$ is continuous in λ_0 (but not constant)

EXAMPLE

$H = L^2[0,1]$ multiplication operator $T|\psi\rangle = x|\psi\rangle$

$$E(\lambda)|\psi\rangle = \begin{cases} \psi(x) & x \le \lambda \\ 0 & x > \lambda \end{cases} = \theta(\lambda - x)\,\psi(x)$$

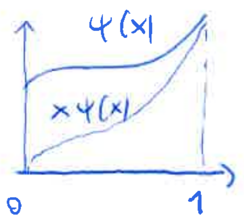
$E(\lambda)$ has no step

$\Rightarrow$ no eigenvalue

Equivalently, suppose λ is an eigenvalue and ψ_λ the corresponding eigenvector

$$T|\psi_\lambda\rangle = \lambda|\psi_\lambda\rangle$$

$$x\,\psi_\lambda(x) = \lambda\,\psi_\lambda(x)$$



$\Rightarrow$ they cannot be proportional!

$\Rightarrow$ T has no eigenvalue

Actually it is possible to define eigenvectors by considering $\psi(x)$ localized in a point

$\psi(x) \sim \delta(x-\lambda)$ $\Rightarrow$ $T\psi(x) = \lambda\,\delta(x-\lambda)$

but $\delta(x-\lambda) \notin L^2[0,1]$ it is a distribution!

$\Rightarrow$ $\lambda \in [0,1]$ $\lambda \in \sigma_c(T)$

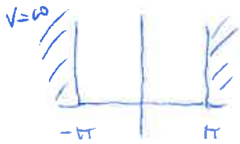
QUANTUM MECHANICS

A physical system has a state which is described as an element of a Hilbert space H

$|\psi\rangle \in H \qquad \|\psi\| = 1$

example

- Spin ½ particle $\quad H = \mathbb{C}^2 \quad |\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2 \quad |a|^2 + |b|^2 = 1$
- particle in 1 dimension $\quad H = L^2(-\infty, \infty) \quad \psi(x)$ WAVE FUNCTION
- particle in a potential well $\quad H = L^2[-\pi, \pi] \quad \psi(x)$



Observables are described as Hermitian operators (you can measure them simultaneously only if they COMMUTE:) $\quad [X, P_x] = i\hbar$

example

- z component of ½ spin $\quad s_z = \frac{1}{2}\sigma_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \qquad s_z^+ = s_z$

$s_i = \frac{1}{2}\sigma_i$

- Position operator $\quad Q|\psi\rangle = x|\psi\rangle \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}$
- Momentum operator $\quad P|\psi\rangle = -i\hbar \frac{\partial}{\partial x}\psi(x) \qquad \Rightarrow \frac{1}{4}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) = \frac{3}{4}I$

If the system is in a state $|\psi\rangle$ and a measure is done of the observable O

$\Rightarrow$ the expectation value of the observable is

$$\langle\psi|O|\psi\rangle = \langle\psi|\sum_i \lambda_i P_i|\psi\rangle = \sum_i \lambda_i \underbrace{\langle\phi|P_i|\psi\rangle}_{= w(\lambda_i)}$$

$\subset$ if $\sigma(O) = \sigma_d(O)$

$$\sum_i w(\lambda_i) = \sum_i \langle\psi|P_i|\psi\rangle = \langle\psi|\mathbb{1}|\psi\rangle = \|\psi\|^2 = 1$$

- If the spectrum of O is a pure discrete spectrum we will always get an eigenvalue
- We cannot say which one, but only its probability!
- If $O|\psi\rangle = \lambda_i|\psi\rangle \quad \Rightarrow \quad w(\lambda_i) = 1$ and $w(\lambda_j) = 0 \quad j \neq i$

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example $\quad S_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \quad |\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \quad P_{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad P_{-\frac{1}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

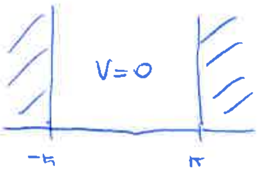
$w(\tfrac{1}{2}) = \langle\psi| P_{\frac{1}{2}} |\psi\rangle = |a|^2$

$w(-\tfrac{1}{2}) = \langle\psi| P_{-\frac{1}{2}} |\psi\rangle = |b|^2$

$\langle\psi| S_z |\psi\rangle = \frac{1}{2}|a|^2 - \frac{1}{2}|b|^2$

expectation value of a measurement of S_z

example



$\psi_m = \frac{1}{\sqrt{2\pi}} e^{imx} \quad m \in \mathbb{N} \quad \langle\psi_n|\psi_m\rangle = \delta_{nm}$

$$P|\psi_m\rangle = -i\hbar \frac{\partial}{\partial x}|\psi_m\rangle = -i\hbar \frac{1}{\sqrt{2\pi}} im\, e^{imx} = m\hbar|\psi_m\rangle$$

- P is not bounded but it has a compact resolvent $\Rightarrow \sigma(\hat{P}) = \sigma_d(\hat{P}) = \{\hbar m, m \in \mathbb{N}\}$

$$\langle\psi|\hat{P}|\psi\rangle = \int_{-\pi}^{\pi} \psi^*(x)\left(-i\hbar \frac{d}{dx}\right)\psi(x)\, dx = \sum \hbar m\, w(m)$$

$$\hat{P} = \sum_{m \in \mathbb{N}} \hbar m\, |\psi_m\rangle\langle\psi_m|$$

projectors

$$w(m) = \langle\psi|P_m|\psi\rangle = |\langle\psi|\psi_m\rangle|^2 = \left| \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} e^{-imx}\psi(x)\,dx \right|^2$$

- $\hat{X}$ position operator $\nexists$ eigenstate, pure continuous spectrum $\sigma(X) = \sigma_c(X) = [-\pi, \pi]$

bounded, but not compact, and no compact resolvent