1)

INTEGRAL EQUATIONS

Relate the unknown function not only to its values at neighborhood points, like differential equations, but to its values throughout a region (boundary conditions are built in!)

We consider $y \in C[a,b]$ and $k(t,s)$ continuous over $[a,b]\times[a,b]$

We look for $x(t)$ such that

$$\int_a^t k(t,s)\,x(s)\,ds = y(t)$$ Volterra eq. of the 1st kind

$$\int_a^b k(t,s)\,x(s)\,ds = y(t)$$ Fredholm eq. of the 1st kind

$$x(t) = y(t) + \mu \int_a^t k(t,s)\,x(s)\,ds$$ Volterra eq. of the 2nd kind

$$x(t) = y(t) + \mu \int_a^b k(t,s)\,x(s)\,ds$$ Fredholm eq. of the 2nd kind

$K : x(t) \to \int_a^{b(t)} k(t,s)\,x(s)\,ds$ is an integral operator. It maps

$L^2[a,b] \to C[a,b]$ and it is compact (proof requires Ascoli-Arzelà theorem)

Note that K is symmetric and Hermitian if $k(t,s) = k^*(s,t)$

We will focus on the Volterra and Fredholm equations of the 2nd kind

They can be written as

$$(1-\mu K)|x\rangle = |y\rangle \quad \text{or} \quad \left(K - \frac{1}{\mu}\right)|x\rangle = -\frac{1}{\mu}|y\rangle \equiv |y'\rangle$$

since K is compact we can use what we have learnt about compact operators.

2)

Voltena equation of the 2nd kind

Theorem: the operator $1-\mu K$ can always be inverted ($\mu \neq 0$)

$\Rightarrow$ the Voltena eq. of the second kind has a unique solution

Proof: Let $M = \max\limits_{[a,b]\times[a,b]} |k(s,t)|$ $\Rightarrow$ we can show by induction that

$$\|K^n x\| \leq M^n \|x\| \frac{(b-a)^n}{n!} \qquad \text{where } \|x\| = \max_{a\leq t\leq b} |x(t)|$$

We have
$$\left| \int_a^t k(t,s)\, x(s)\, ds \right| \leq M \int_a^t |x(s)|\, ds \leq M \|x\| \,(t-a)$$

$\Rightarrow$ the above inequality holds for $n=1$

Suppose that for $n-1$ $\quad |K^{n-1}x|(t) \leq M^{n-1} \|x\| \dfrac{(t-a)^{n-1}}{(n-1)!}$

$$\Rightarrow \quad |K^n x|(t) = \left| \int_a^t k(t,s)\,(K^{n-1}x)(s)\, ds \right| \leq \frac{M^{n-1}\|x\|}{(n-1)!} \int_a^t M\,(s-a)^{n-1}\, ds$$

$$= \frac{M^n}{n!} \|x\| \,(t-a)^n$$

$$\Rightarrow \quad \|K^n x\| \leq M^n \|x\| \frac{(b-a)^n}{n!}$$

But now sending $K \to \mu K$ we have $\|\mu K\|^n \leq |\mu|^n M^n \dfrac{(b-a)^n}{n!}$

$$\left\| \sum_{n=0}^{N} (\mu K)^n \right\| \leq \sum_{n=0}^{N} \|(\mu K)^n\| \leq \sum_{n=0}^{N} |\mu|^n M^n \frac{(b-a)^n}{n!} \to e^{M|\mu|(b-a)}$$

$\Rightarrow$ the series converges, and we can sum it treating it as a geometrical series

$\Rightarrow$ the series converges to $(1-\mu K)^{-1}$

3)

<u>Fredholm eq. of the 2nd kind</u> (we limit ourselves, for simplicity, to the case $K = K^+$)
i.e. $k(s,t) = k^*(t,s)$

We can write the equation in the form $(\mathbb{1} - \mu k)|x\rangle = |y\rangle$

$\Rightarrow$ by using the spectral theorem for compact operators we can state that

- if $\lambda = \frac{1}{\mu}$ is <u>not</u> an eigenvalue of k $\Rightarrow$ a solution of the Fredholm equation always exists
- if $\lambda = \frac{1}{\mu}$ <u>is</u> an eigenvalue $\Rightarrow$ a solution exists if $y \in H_\lambda^\perp$
(H_λ eigenspace corresponding to λ)

The above alternative goes under the name of <u>Fredholm Alternative</u>

We can write $\quad k - \lambda I = \sum_i (\lambda_i - \lambda)|\varphi_i\rangle\langle\varphi_i|$ spectral decomposition

complete orthonormal basis

$$\Rightarrow \quad (k - \lambda I)^{-1}|y\rangle = \sum_i \frac{|\varphi_i\rangle\langle\varphi_i|y\rangle}{\lambda_i - \lambda}$$

this expression is well defined if $\lambda \neq \lambda_i \ \forall i$

if $\lambda = \lambda_k$ eigenvalue we must have $\langle\varphi_k|y\rangle = 0$

$\Rightarrow$ the general solution of the equation can be written as

$$|x\rangle = \sum_i \frac{|\varphi_i\rangle\langle\varphi_i|y\rangle}{\lambda_i - \lambda} \qquad \lambda \text{ not eigenvalue}$$

$$|x\rangle = \sum_{i \neq k} \frac{|\varphi_i\rangle\langle\varphi_i|y\rangle}{\lambda_i - \lambda} + |y_k\rangle \qquad \lambda = \lambda_k \text{ eigenvalue}$$

arbitrary vector in the eigenspace of λ_k

Note that, since the eigenvalues of k fulfill $|\lambda| \leq \|T\|$, if the equation is such that $|\lambda| = \frac{1}{|\mu|} > \|k\|$ $\Rightarrow$ λ cannot be eigenvalue!

EXAMPLE

$H = L^2[0,1]$ $\qquad (Tx)(t) = \int_0^1 st\, x(s)\,ds$

eigenvalues : $(Tx)(t) = t\int_0^1 s\, x(s)\,ds = \lambda x(t) \Rightarrow$ for $\lambda \neq 0$ we must have $x(t) = ct$

$$t\int_0^1 s\,\cancel{c}\,s\,ds = \lambda\,\cancel{c}\,t \Rightarrow \lambda_1 = \frac{1}{3}$$

All the other eigenvectors correspond to $\lambda = 0$ and are such that $\int_0^1 s\,x(s)\,ds = 0$

We now want to discuss the spectral decomposition of T

and the resolvent $R_T(z)$ $z \in \mathbb{C}$

To do this, we have to find a basis for $L^2[0,1]$ such that a suitably normalized $|\varphi_1\rangle$ corresponds to $\lambda_1 = \frac{1}{3}$ and is a basis vector.

We observe that $P_\ell(2t-1)$ form a basis for $L^2[0,1]$

$\Rightarrow$ we can choose

$$\tilde{\varphi}_1(t) = P_0(2t-1) = 1$$
$$\tilde{\varphi}_2(t) = P_1(2t-1)\cdot\sqrt{3} = \sqrt{3}(2t-1)$$
$$\dots$$
$$\tilde{\varphi}_m(t) = \sqrt{2m-1}\; P_{m-1}(2t-1)$$

The desired vector $|\varphi_1\rangle$ is neither $|\tilde{\varphi}_1\rangle$ nor $|\tilde{\varphi}_2\rangle$ but a linear combination of them : $|\varphi_1\rangle = \sqrt{3}\,t$. We can then define the orthonormal combination $|\varphi_2\rangle = 3t-2$ and $|\varphi_m\rangle = |\tilde{\varphi}_m\rangle$ for $m > 2$.

The spectral decomposition of T is thus

$$T = \frac{1}{3}|\varphi_1\rangle\langle\varphi_1| \qquad \text{and}$$

$$R_T(z) = \sum_i \frac{1}{\lambda_i - z}|\varphi_i\rangle\langle\varphi_i| = \frac{1}{\frac{1}{3}-z}|\varphi_1\rangle\langle\varphi_1| - \frac{1}{z}\sum_{i=2}^{\infty}|\varphi_i\rangle\langle\varphi_i|$$

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- Let us now consider the Fredholm equation

$$x(t) - 4\int_0^1 (st)\, x(s)\, ds = 6t(2-t)$$ which can be rewritten as

$$(1-4T)|x\rangle = |y\rangle \qquad \lambda_i \neq \frac{1}{4} \implies \exists!\ \text{solution}$$

$$|x\rangle = -\frac{1}{4}\left(T - \frac{1}{4}I\right)^{-1}|y\rangle = -\frac{1}{4}\left(\frac{1}{\frac{1}{3}-\frac{1}{4}}|\varphi_1\rangle\langle\varphi_1|y\rangle - 4\sum_{i=2}^{\infty}\langle\varphi_i|y\rangle|\varphi_i\rangle\right)$$

$$= -3\langle\varphi_1|y\rangle|\varphi_1\rangle + \sum_{i=2}^{\infty}\langle\varphi_i|y\rangle|\varphi_i\rangle \qquad \left(\varphi_3 = \frac{\sqrt{5}}{2}\left(3(2t-1)^2-1\right)\right)$$

$\implies$ computing the scalar products we get $|x\rangle = -6t(t+3)$

- Let us now consider the equation

$$x(t) - 3\int_0^1 (st)\, x(s)\, ds = 6t(k-t)$$

for which values of the parameter k has the equation solution?

Since $z = \frac{1}{3}$ is eigenvalue, we must check that $\langle\varphi_1|y\rangle = 0$

$$\langle\varphi_1|y\rangle = \int_0^1 \sqrt{3}\,t\ 6t(k-t)\,dt = 6\sqrt{3}\left(\frac{k}{3} - \frac{1}{4}\right) \implies k = \frac{3}{4}$$