Precision determination(s) of α_s from lattice QCD

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work done in collaboration with

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ALPHA Collaboration

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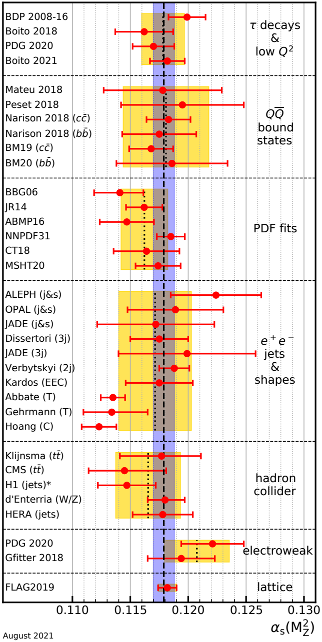
Current situation for α_s

- α_s is a **fundamental** parameter of the SM
- Impacts virtually all theoretical calculations for x-sections & decays for LHC
- Relevant also for EW vacuum stability, GUT, & searches of new colored sectors
- **PDG: $\alpha_s(m_Z) = 0.1179(9) \approx 0.8\%$**
  Not good enough! We want $\ll 1\%$, else
  - $\Rightarrow$ Large uncertainties in key processes (Higgs)
  - $\Rightarrow$ Limiting factor for precision top mass and EWPO determinations at future colliders
- Many determinations are precision limited by **systematics**: PT truncation errors, non-pert. effects, ...
- Lattice QCD is a **powerful** tool for the job

(PDG '21)

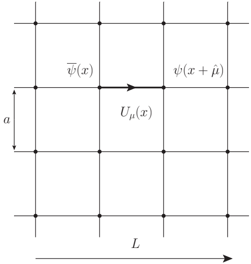


Crash course in lattice QCD

(Wilson '74; ...)

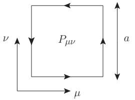
Path integral

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int DA D\psi D\overline{\psi}\, \mathcal{O}[A,\psi,\overline{\psi}]\, e^{-S_{\mathrm{QCD}}[A,\psi,\overline{\psi}]}$$



Gauge action

$$S_G = \frac{1}{g_0^2} \sum_{x,\mu,\nu} \mathrm{Re\,tr}\{1 - P_{\mu\nu}(x)\}$$



Fermion action

$$S_F = a^4 \sum_{f=1}^{N_f} \sum_x \overline{\psi}_f(x) \overbrace{(D_{\mathrm{w}} + m_{0,f})}^{D_f} \psi_f(x) \qquad D_{\mathrm{w}} = \frac{1}{2} \sum_\mu \{\gamma_\mu(\nabla_\mu^\star + \nabla_\mu) - a\nabla_\mu^\star \nabla_\mu\}$$

- ✓ Theoretically robust and cheap to simulate
- ✗ Hard breaking of $\mathrm{SU_A}(N_f)$ symmetry for $m_{0,f} = 0$

Continuum limit, $a \to 0$

$$g_0^2(a) \to 0 \qquad a \equiv \frac{(am_p)}{m_p^{\mathrm{exp}}} \qquad \frac{(am_{\mathrm{had}})}{(am_p)} = \frac{m_{\mathrm{had}}^{\mathrm{exp}}}{m_p^{\mathrm{exp}}} \qquad \mathrm{had} = \pi, K, \ldots \;\Rightarrow\; m_{0,f}(a)$$

Infinite volume limit, $L \to \infty$

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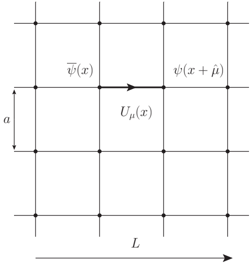

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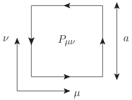

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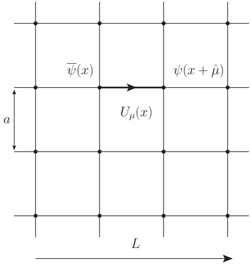

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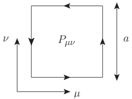

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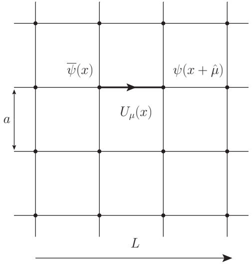

Gauge action

$$S_G \overset{a\to 0}{\approx} \frac{1}{4g_0^2} \int \mathrm{d}^4x\, F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) \qquad U_\mu(x) \overset{a\to 0}{\approx} e^{iaA_\mu(x)}$$

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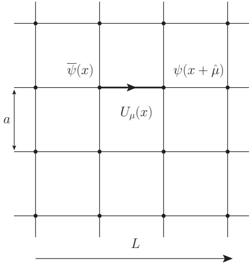

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α_s from lattice QCD

All there is to it

$$\mathcal{O}(q) \overset{q\to\infty}{\approx} \sum_{n=1}^{N} c_n \alpha^n_{\overline{\rm MS}}(q) + \mathrm{O}(\alpha^{N+1}_{\overline{\rm MS}}) + \mathrm{O}\left(\frac{\Lambda^p}{q^p}\right) \qquad \left[\alpha_{\mathcal{O}}(q) \equiv \frac{\mathcal{O}(q)}{c_1}\right]$$

Why do we like it?

- Lots of freedom in choosing $\mathcal{O}$ $\Rightarrow$ no need to be exp. accessible
- $\mathcal{O}$ defined within QCD $\Rightarrow$ EW effects only affect hadronic inputs
- $\mathcal{O}(q)$ non-pert. and accurately measurable up to large scales q [**if carefully chosen**]
- No need for modeling hadronization

It all starts at low-energy

Lattice QCD parameters are renormalized (fixed) in terms of hadronic inputs

$$f_\pi,\ \underbrace{m_\pi,\ m_K,\ \ldots}_{N_{\rm f}} \quad \Rightarrow \quad g_0,\ \underbrace{m_{0,ud},\ m_{0,s},\ \ldots}_{N_{\rm f}}$$

QCD coupling and quark masses in any other **scheme**, at **any scale**, are **predictions**

Caveat

In most calculations $N_{\rm f} = 3$. What happens with the charm and bottom? Later!

Meet the challenge

LQCD butchers space-time by introducing

1. **Lattice spacing** a, i.e. UV-cutoff $\sim a^{-1}$
2. **Finite volume** L^4, i.e. IR-cutoff $\sim L^{-1}$

Systematic error constraints

- **Low-energy**: hadronic inputs $m_{\rm had}$

$$L^{-1} \ll m_{\rm had} \ll a^{-1} \qquad m_{\rm had} \overset{\text{e.g.}}{=} f_\pi, m_\pi, m_K, \ldots \sim \Lambda_{\rm QCD}$$

- **High-energy:** non-pert. coupling $\alpha_{\mathcal{O}}(q)$

$$L^{-1} \ll q \ll a^{-1} \qquad q \gg \Lambda_{\rm QCD}$$

Problem

Fitting hadronic and pQCD scales into a single lattice requires

$$L^{-1} \ll m_{\rm had} \ll q \ll a^{-1}$$

- Most common lattice simulations are devised for $m_{\rm had}$ calculations
- Cost of simulations $\propto (L/a)^{-7} \Rightarrow q \times 2$ is $\mathrm{O}(100)\times$ more costly
- $\alpha_{\mathcal{O}}(q) \overset{q\to\infty}{\propto} 1/\log(q/\Lambda_{\rm QCD}) \Rightarrow$ **Exponentially HARD problem!**

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Problem

Fitting hadronic and pQCD scales into a single lattice requires

$$L/a \sim 100 \qquad m_\pi L \sim 4 \quad \Rightarrow \; a^{-1} \sim 3\,\text{GeV} \;\Rightarrow\; q \sim \text{O}(1)\,\text{GeV}$$

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How can we reach high-energy?

Computations of $m_{\rm had}$ and $\alpha_{\mathcal{O}}(q)$ are separate problems

$\Rightarrow$ precision demands **dedicated** approach for $\alpha_{\mathcal{O}}(q)$

Finite-volume schemes

(Wilson; …; Lüscher, Weisz, Wolff '92)

- Finite-L effects are part of the *definition of* $\alpha_{\mathcal{O}}(q)$, i.e. $q = L^{-1}$
  Measure the change in finite-volume correlators as L varies
- Lattice systematics are under control once

$$L^{-1} = q \ll a^{-1} \;\Rightarrow\; L/a \gg 1 \;\Rightarrow\; \textbf{EASY!}$$

Step-scaling strategy

(Lüscher et al. '94; Jansen et al. '96)

1. Given $\alpha_{\mathcal{O}}(q_{\rm had} = L_{\rm had}^{-1}) \overset{\text{e.g.}}{=} 1$, determine $q_{\rm had}/m_{\rm had} \sim {\rm O}(1)$
2. Measure change in $\alpha_{\mathcal{O}}(q = L^{-1})$ as $L \to L/2$

$$\sigma_{\mathcal{O}}(u) \equiv \alpha_{\mathcal{O}}(2q)|_{u=\alpha_{\mathcal{O}}(q)} \quad \Rightarrow \quad \text{non-pert. } \beta\text{-function}$$

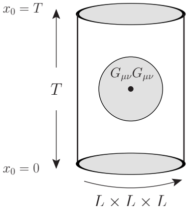
3. Starting from $q_{\rm had} \sim \Lambda_{\rm QCD}$, after $n \sim {\rm O}(10)$ steps, we reach

   $q_{\rm PT} = 2^n q_{\rm had} \sim {\rm O}(100)\,{\rm GeV}$ where $\alpha_{\mathcal{O}}(q_{\rm PT}) \sim 0.1$
4. Extract $\alpha_{\overline{\rm MS}}(q_{\rm PT})$ from PT expansion of $\alpha_{\mathcal{O}}(q_{\rm PT})$
5. $\alpha_{\overline{\rm MS}}(q_{\rm PT}) \overset{\rm PT}{\to} \Lambda_{\overline{\rm MS}}/q_{\rm PT} \to \Lambda_{\overline{\rm MS}}/q_{\rm had} \to \Lambda_{\overline{\rm MS}}/m_{\rm had}$

Schrödinger functional couplings

(Symanzik '81; Lüscher et al. '92; Sint '94; ...)

Gauge fields bcs.



$$A_k(x)|_{x_0=0} = C_k(\eta,\nu) \qquad A_k(x)|_{x_0=T} = C'_k(\eta,\nu)$$

Quark fields bcs. $\left[\, P_\pm = \frac{1}{2}(1 \pm \gamma_0) \,\right]$

$$P_+\psi|_{x_0=0} = P_-\psi|_{x_0=T} = 0$$

$$\overline{\psi}P_-|_{x_0=0} = \overline{\psi}P_+|_{x_0=T} = 0$$

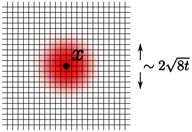
SF coupling

$$\alpha_{\mathrm{SF},\nu}(q) \propto \left.\frac{1}{\partial_\eta \Gamma}\right|_{\eta=0} \qquad \Gamma = -\ln \mathcal{Z}[C, C'] \qquad q = L^{-1} \qquad \overline{m} = 0$$

Gradient flow (GF)

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) \qquad B_\mu(0,x) = A_\mu(x)$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \qquad D_\mu = \partial_\mu + [B_\mu, \cdot\,]$$



Gauge-invariant composite fields of B_μ are **finite** for $t > 0$ (Lüscher, Weisz '12)

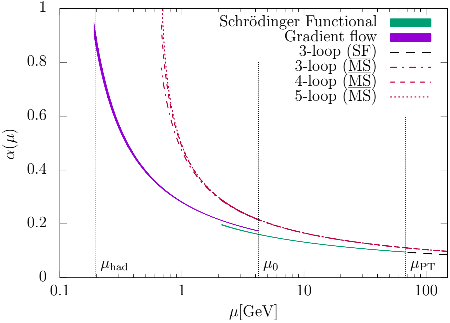
GF coupling

$$\alpha_{\mathrm{GF}}(q) \propto t^2 \langle G^a_{\mu\nu}(t,x) G^a_{\mu\nu}(t,x)\rangle|_{x_0=T/2} \qquad q = L^{-1} \qquad \sqrt{8t} = L/3 \qquad \overline{m} = 0$$

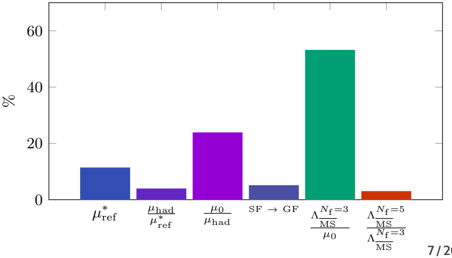
α_s from a non-perturbative determination of $\Lambda_{\overline{\rm MS}}^{(N_f=3)}$

1. Determination of $\mu_{\rm had}/f_{\pi,K}$ to establish $\mu_{\rm had} = 197(3)\,{\rm MeV}$ where $\alpha_{\rm GF}^{(3)}(\mu_{\rm had}) = 0.9$
2. Non-pert. running GF-scheme from $\mu_{\rm had}$ to $\mu_0 = 4.3(1)\,{\rm GeV}$
3. Non-pert. matching finite-volume schemes: GF $\to$ SF
4. Non-pert. running SF-scheme from μ_0 to $\mu_{\rm PT} = 2^4\mu_0 \sim 70\,{\rm GeV}$
5. NNLO matching SF $\to \overline{\rm MS}$ schemes and $\alpha_{\overline{\rm MS}}^{(3)}(\mu_{\rm PT})$ extraction
6. $\alpha_{\overline{\rm MS}}^{(3)}(\mu_{\rm PT}) \to \Lambda_{\overline{\rm MS}}^{(3)} = \underbrace{341(12)\,{\rm MeV}}_{3.5\%}$
7. PT decoupling for c- and b-quarks gives $\Lambda_{\overline{\rm MS}}^{(3)} \to \Lambda_{\overline{\rm MS}}^{(5)} \to \alpha_{\overline{\rm MS}}^{(5)}(m_Z) = \underbrace{0.1185(8)}_{0.7\%}$

(ALPHA Collab. '17)



Contribution to relative error squared of α_s



How accurate is $N_\mathrm{f} = 3$ QCD?

Including the charm quark in hadronic simulations is challenging

- **Very fine** lattice spacings are needed $\Rightarrow$ **CPU expensive**
  $m_c \sim 1.3\,\mathrm{GeV} \;\Rightarrow\; am_c \gtrsim 0.3$ in typical simulations
- More costly simulations and complex tuning of parameters
  $g_0, m_{0,ud}, m_{0,s}, m_{0,c} \;\Leftrightarrow\; f_\pi, m_\pi, m_K, m_D$

Systematics in $\Lambda^{(3)}_{\overline{\mathrm{MS}}} \to \Lambda^{(5)}_{\overline{\mathrm{MS}}}$

- **Matching Λ-parameters**
  The ratios $\Lambda^{(3)}_{\overline{\mathrm{MS}}}/\Lambda^{(4)}_{\overline{\mathrm{MS}}}$ and $\Lambda^{(4)}_{\overline{\mathrm{MS}}}/\Lambda^{(5)}_{\overline{\mathrm{MS}}}$ are given by

$$P_{\ell,\mathrm{f}}(M/\Lambda^{(N_\mathrm{f})}_{\overline{\mathrm{MS}}}) = \Lambda^{(N_\ell)}_{\overline{\mathrm{MS}}}/\Lambda^{(N_\mathrm{f})}_{\overline{\mathrm{MS}}} \qquad M \equiv \text{RGI-mass decoupling quark(s)}$$

- **Hadronic quantities**
  Renormalization of lattice QCD requires tuning $g_0, m_{0,ud}, \ldots$, so that

$$R_\mathrm{had} \overset{\text{e.g.}}{=} \left[\frac{m_\pi}{f_\pi}\right]^\mathrm{lat}, \left[\frac{m_K}{f_\pi}\right]^\mathrm{lat}, \ldots = \left[\frac{m_\pi}{f_\pi}\right]^\mathrm{exp}, \left[\frac{m_K}{f_\pi}\right]^\mathrm{exp}, \ldots$$

  $m^\mathrm{exp}_\mathrm{had} \equiv$ exp. value (corrected for QED and $m_u \neq m_d$ effects)

  Q: What's the size of charm effects: $R^{(N_\mathrm{f}=3)}_\mathrm{had} = R^{(N_\mathrm{f}=4)}_\mathrm{had} + \mathrm{O}(M_c^{-2})$?

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$$P_{\ell,\mathrm{f}}(M/\Lambda^{(N_\mathrm{f})}_{\overline{\mathrm{MS}}}) \sim P^{(n\text{-loop})}_{\ell,\mathrm{f}}(M/\Lambda^{(N_\mathrm{f})}_{\overline{\mathrm{MS}}}) + \mathrm{O}(\alpha^{n-1}(M)) + \mathrm{O}(M^{-2})$$

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Effective theory of decoupling and PT matching

Fundamental theory

$$\mathcal{L}_{\mathrm{QCD}_{\mathrm{N_f}}} = \frac{1}{4g^2} F_{\mu\nu} F_{\mu\nu} + \sum_{f=1}^{N_\ell} \overline{\psi}_f \not{D} \psi_f + \sum_{f=N_\ell+1}^{N_\mathrm{f}} \overline{\psi}_f (\not{D} + M) \psi_f$$

Effective theory

(Weinberg '80; ...)

$$\mathcal{L}_{\mathrm{eff}} = \mathcal{L}_{\mathrm{QCD}_{\mathrm{N}_\ell}} + \frac{1}{M^2} \sum_i \omega_i \Phi_i + \ldots \quad \Rightarrow \quad \mathbf{LO:} \quad \mathcal{L}_{\mathrm{eff}} = \mathcal{L}_{\mathrm{QCD}_{\mathrm{N}_\ell}}$$

Matching couplings in PT

(Bernreuther, Wetzel '82; ...; Chetyrkin, Kühn, Sturm '06; Schröder, Steinhauser '06)

EFT is matched at LO once the effective and fundamental couplings are matched

$$\alpha_{\overline{\mathrm{MS}}}^{(N_\ell)}(m_\star) \equiv \alpha_\star \, \xi(\alpha_\star) \qquad \alpha_\star \equiv \alpha_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}(m_\star) \qquad m_\star = \overline{m}_{\overline{\mathrm{MS}}}(m_\star)$$

Matching Λ-parameters in PT

$$\Lambda_{\overline{\mathrm{MS}}}^{(N_\ell)}(M, \Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}) = P_{\ell,\mathrm{f}}(M/\Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}) \, \Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})} \quad \Rightarrow \quad P_{\ell,\mathrm{f}}(M/\Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}) = \frac{\varphi_{\overline{\mathrm{MS}}}^{(N_\ell)}\big(\alpha_\star \xi(\alpha_\star)\big)}{\varphi_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}(\alpha_\star)}$$

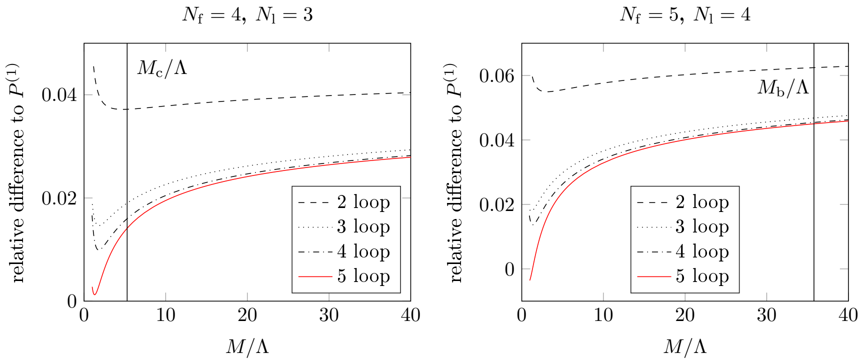
where

$$\Lambda_{\mathrm{X}}^{(N_\mathrm{f})} = \mu \, \varphi_{\mathrm{X}}^{(N_\mathrm{f})}(\alpha_{\mathrm{X}}(\mu)) \qquad \varphi_{\mathrm{X}}^{(N_\mathrm{f})}(\alpha) = \ldots \exp\left\{ - \int_0^\alpha \frac{\mathrm{d}y}{\beta_{\mathrm{X}}^{(N_\mathrm{f})}(y)} + \ldots \right\}$$

$$M = \overline{m}_{\mathrm{X}}(\mu) \, \varepsilon_{\mathrm{X}}^{(N_\mathrm{f})}(\alpha_{\mathrm{X}}(\mu)) \qquad \varepsilon_{\mathrm{X}}^{(N_\mathrm{f})}(\alpha) = \ldots \exp\left\{ - \int_0^\alpha \mathrm{d}y \frac{\tau_{\mathrm{X}}^{(N_\mathrm{f})}(y)}{\beta_{\mathrm{X}}^{(N_\mathrm{f})}(y)} + \ldots \right\}$$

Perturbative decoupling at work

(Athenodorou et al. '18)



- $P_{\ell,\mathrm{f}}(M/\Lambda) \sim P_{\ell,\mathrm{f}}^{(n\text{-loop})}(M/\Lambda) + \mathrm{O}(\alpha_\star^{n-1})$
- PT expansion shows **very good** "convergence"

$\Rightarrow$ PT uncertainties are quite **small**

Q: But can we really trust PT decoupling at M_c/Λ?

n-loop	$\alpha_{\overline{\mathrm{MS}}}^{(5)}(m_Z)$	$\alpha_n - \alpha_{n-1}$
2	0.11699	
3	0.11827	0.00128
4	0.11846	0.00019
5	0.11852	0.00006

$$\alpha_{\overline{\mathrm{MS}}}^{(5)}(m_Z) = 0.1185(8)(3)_{\mathrm{PT}}$$

How perturbative are heavy quarks?

Non-perturbative matching

$$\frac{\Lambda^{(N_\ell)}}{m_{\mathrm{had},1}^{(N_\ell)}} = P_{\ell,\mathrm{f}}^{\mathrm{had},1}\left(M/\Lambda^{(N_\mathrm{f})}\right)\frac{\Lambda^{(N_\mathrm{f})}}{m_{\mathrm{had},1}^{(N_\mathrm{f})}(M)} \quad \Rightarrow \quad m_{\mathrm{had},2}^{(N_\ell)} = m_{\mathrm{had},2}^{(N_\mathrm{f})}(M) + \mathrm{O}\left(\frac{\Lambda^2}{M^2}\right)$$

Factorization formula

(Bruno et al. '15; Athenodorou et al. '18)

$$\frac{m_{\mathrm{had}}^{(N_\mathrm{f})}(M)}{m_{\mathrm{had}}^{(N_\mathrm{f})}(0)} = \mathcal{Q}_{\ell,\mathrm{f}}^{\mathrm{had}} \times P_{\ell,\mathrm{f}}^{\mathrm{had}}\left(M/\Lambda^{(N_\mathrm{f})}\right) = \underbrace{\mathcal{Q}_{\ell,\mathrm{f}}^{\mathrm{had}}}_{\text{NP \& } M\text{-indep.}} \times \underbrace{P_{\ell,\mathrm{f}}\left(M/\Lambda^{(N_\mathrm{f})}\right)}_{\text{PT \& universal}} + \mathrm{O}\left(\frac{\Lambda^2}{M^2}\right)$$

Result: Typical $\mathrm{O}(\Lambda^2/M_c^2)$ corrections to $P_{3,4}(M_c/\Lambda)$ are $< 1\%$ effects (Athenodorou et al. '18)

$\Rightarrow \Lambda_{\overline{\mathrm{MS}}}^{(3)} \stackrel{\mathrm{PT}}{\rightarrow} \Lambda_{\overline{\mathrm{MS}}}^{(4)}$ **precise** enough for $\delta\Lambda_{\overline{\mathrm{MS}}}^{(3)} \gtrsim 1.5\%$

Ratios of hadronic scales

$$\frac{m_{\mathrm{had},1}^{(N_\mathrm{f})}(M)}{m_{\mathrm{had},2}^{(N_\mathrm{f})}(M)} = \frac{m_{\mathrm{had},1}^{(N_\ell)}}{m_{\mathrm{had},2}^{(N_\ell)}} + \mathrm{O}\left(\frac{\Lambda^2}{M^2}\right)$$

Result: Typical $\mathrm{O}(\Lambda^2/M_c^2)$ corrections to such ratios are $< 0.5\%$ effects

$\Rightarrow$ Good enough for a **per-cent** precision determination of $\Lambda_{\overline{\mathrm{MS}}}^{(3)}$ (Knechtli et al. '17; Höllwieser et al. '20)

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Non-perturbative renormalization by decoupling

Current situation

- $\delta\Lambda_{\overline{\rm MS}}^{(3)} \sim 3.5\% \Rightarrow$ room for **improvement!**
- $\delta\Lambda_{\overline{\rm MS}}^{(3)}$ dominated by NP running $0.2 - 70\,\mathrm{GeV}$
- Halving $\delta\Lambda_{\overline{\rm MS}}^{(3)}$ by brute force is CPU expensive

Key observations

- $P_{\ell,\mathrm{f}}(M/\Lambda)$ has **small** PT and NP corrections for $M/\Lambda \gtrsim 5$
- $\Lambda_{\overline{\rm MS}}^{(N_\mathrm{f})}$ is M-independent $\Rightarrow$ same for $\mathrm{QCD}_{N_\mathrm{f}}$ with any M
- LQCD can **access** $\mathrm{QCD}_{N_\mathrm{f}}$ with any M

Master equation 1.0

(ALPHA Collab. '20, '22)

$$\frac{\Lambda^{(N_\ell)}}{m_{\rm had}^{(N_\ell)}} = P_{\ell,\mathrm{f}}^{\rm had}\left(M/\Lambda^{(N_\mathrm{f})}\right)\frac{\Lambda^{(N_\mathrm{f})}}{m_{\rm had}^{(N_\mathrm{f})}(M)}$$

- Compute $\Lambda_{\overline{\rm MS}}^{(0)}/m_{\rm had}^{(0)}$ in **pure Yang-Mills**
- Determine $m_{\rm had}^{(3)}(M)/m_{\rm had}^{(3)}(m_{u,d,s}^{\rm phys})$ and set $m_{\rm had}^{(3)}(m_{u,d,s}^{\rm phys}) \equiv m_{\rm had}^{\rm exp}$
- Extrapolate for $M \to \infty$

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Master equation 1.0

(ALPHA Collab. '20, '22)

$$\frac{\Lambda^{(0)}_{\overline{\rm MS}}}{m^{(0)}_{\rm had}} = P^{(n\text{-loop})}_{0,3}\left(M/\Lambda^{(3)}_{\overline{\rm MS}}\right)\frac{\Lambda^{(3)}_{\overline{\rm MS}}}{m^{(3)}_{\rm had}(M)} + \mathrm{O}(\alpha_\star^{n-1}) + \mathrm{O}\left(\frac{\Lambda^2}{M^2}\right)$$

- Compute $\Lambda^{(0)}_{\overline{\rm MS}}/m^{(0)}_{\rm had}$ in **pure Yang-Mills**
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- Extrapolate for $M \to \infty$

Non-perturbative renormalization by decoupling

Is this feasible?

$$L^{-1} \ll m_{\mathrm{had}}^{(3)} \ll M \ll a^{-1}$$

Example

$$L/a = 100 \qquad m_\pi L \sim 4 \quad \Rightarrow \; a^{-1} \sim 3\,\mathrm{GeV} \; \Rightarrow \; M \sim 1\,\mathrm{GeV}$$

Decoupling in a finite volume

- **Decoupling scale**

$$\alpha_{\mathrm{GF}}^{(3)}(\mu_{\mathrm{dec}}^{(3)}) = 0.3 \quad \Rightarrow \quad \mu_{\mathrm{dec}}^{(3)} = L_{\mathrm{dec}}^{-1} = 789(15)\,\mathrm{MeV}$$

- **Massive coupling** (Appelquist, Carazzone '75; ...)

$$\alpha_{\mathrm{GF}}^{(0)}(\mu_{\mathrm{dec}}^{(0)}) \overset{\text{def.}}{=} \alpha_{\mathrm{GF}}^{(3)}(\mu_{\mathrm{dec}}^{(3)}, M) \quad \Rightarrow \quad \mu_{\mathrm{dec}}^{(0)} = \mu_{\mathrm{dec}}^{(3)} + \mathrm{O}(M^{-2}) \sim \mu_{\mathrm{dec}}$$

Master formula 2.0 (ALPHA Collab. '20, '22)

$$\frac{\Lambda_{\overline{\mathrm{MS}}}^{(0)}}{\mu_{\mathrm{dec}}} = P_{0,3}^{(n\text{-loop})}(M/\Lambda_{\overline{\mathrm{MS}}}^{(3)}) \frac{\Lambda_{\overline{\mathrm{MS}}}^{(3)}}{\mu_{\mathrm{dec}}} + \mathrm{O}(\alpha_\star^{n-1}) + \mathrm{O}\left(\frac{\mu_{\mathrm{dec}}^2}{M^2}\right)$$

- Determine $\alpha_{\mathrm{GF}}^{(3)}(\mu_{\mathrm{dec}}, M)$ such that $L_{\mathrm{dec}}^{-1} = \mu_{\mathrm{dec}} \ll M \ll a^{-1}$

$$L_{\mathrm{dec}}/a \sim 50 \qquad \mu_{\mathrm{dec}} \sim 800\,\mathrm{MeV} \quad \Rightarrow \quad M \sim 10\,\mathrm{GeV}$$

- Compute $\Lambda_{\overline{\mathrm{MS}}}^{(0)}/\mu_{\mathrm{dec}} = (\Lambda_{\overline{\mathrm{MS}}}^{(0)}/\Lambda_{\mathrm{GF}}^{(0)})\varphi_{\mathrm{GF}}^{(0)}(\alpha_{\mathrm{GF}}^{(0)}(\mu_{\mathrm{dec}}))$

Large-mass limit

Effective action

$$\mathcal{L}_{\mathrm{QCD}} \approx \mathcal{L}_{\mathrm{YM}} + \frac{1}{M^2}\mathcal{L}_{2,\mathrm{dec}} + \dots \qquad \mathcal{L}_{\mathrm{YM}} = \frac{1}{4g^2} F^a_{\mu\nu}F^a_{\mu\nu}$$

$$\langle \mathcal{O}_{\mathrm{GF}}\rangle_{\mathrm{QCD}} = \langle \mathcal{O}_{\mathrm{GF}}\rangle_{\mathrm{YM}} - \frac{1}{M^2}\int \mathrm{d}^4x \langle \mathcal{O}_{\mathrm{GF}}\mathcal{L}_{2,\mathrm{dec}}(x)\rangle^{\mathrm{conn}}_{\mathrm{YM}} + \mathrm{O}(M^{-3})$$

$\mathrm{O}(1/M^2)$ counterterm

$$\mathcal{L}_{2,\mathrm{dec}} = \sum_{i=1}^{2} d_i(g^2)\mathcal{D}_i$$

$$\mathcal{D}_1 = \frac{1}{g^2}\mathrm{tr}\left(D_\mu F_{\mu\nu} D_\rho F_{\rho\nu}\right) \qquad \mathcal{D}_2 = \frac{1}{g^2}\mathrm{tr}\left(D_\mu F_{\rho\nu} D_\mu F_{\rho\nu}\right) - \frac{23}{7}\mathcal{D}_1$$

$\mathrm{O}(1/M^2)$ contribution

$[\ \alpha_\star \equiv \alpha^{(3)}_{\overline{\mathrm{MS}}}(m_\star)\ ]$

$$\alpha^{(3)}_{\mathrm{GF}}(\mu, M) - \alpha^{(0)}_{\mathrm{GF}}(\mu) \propto \frac{1}{M^2}\sum_{i=1}^{2} \alpha_\star^{\hat{\gamma}_i^{\mathcal{D}} - 2\hat{\gamma}_{\mathrm{m}}} d_i(\alpha_\star)\int \mathrm{d}^4x\, \langle \mathcal{O}_{\mathrm{GF}}\mathcal{D}_i^{\mathrm{RGI}}(x)\rangle^{\mathrm{conn}}_{\mathrm{YM}} + \dots$$

- **LO anomalous dim:** $\hat{\gamma}_{\mathrm{m}} = 4/9 \ ; \ \hat{\gamma}_1^{\mathcal{D}} = 0 \ ; \ \hat{\gamma}_2^{\mathcal{D}} = 7/11$ (Husung et al. '20; Husung '21)
- **Matching:** $d_i(\alpha_\star) = \hat{d}_i\alpha_\star + \mathrm{O}(\alpha_\star^2)$

Continuum limit

Symanzik effective action

(Symanzik '82; Sheikholeslami, Wohlert '85; Lüscher et al. '96; …; Husung et al. '22; Husung '23)

$$\mathcal{L}_{\rm latt} \approx \mathcal{L}_{\rm QCD} + \frac{1}{\Lambda_{\rm UV}}\mathcal{L}_1 + \frac{1}{\Lambda_{\rm UV}^2}\mathcal{L}_2 + \ldots \qquad \Lambda_{\rm UV} = a^{-1}$$

$$\langle \mathcal{O}_{\rm GF}\rangle_{\rm latt} = \langle \mathcal{O}_{\rm GF}\rangle_{\rm QCD} - a\int {\rm d}^4x\,\langle \mathcal{O}_{\rm GF}\mathcal{L}_1(x)\rangle_{\rm QCD}^{\rm conn} + {\rm O}(a^2)$$

${\rm O}(a)$ counterterms

$$\mathcal{L}_1 = \sum_{i=1}^{3} c_i(g^2)\mathcal{O}_i \quad \Leftarrow \quad \text{Consequence of breaking } {\rm SU_A}(N_{\rm f}) \text{ symmetry}$$

$$\mathcal{O}_1 = \overline{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi \qquad \mathcal{O}_2 = M^2\overline{\psi}\psi \qquad \mathcal{O}_3 = \frac{M}{4g^2}F^a_{\mu\nu}F^a_{\mu\nu}$$

${\rm O}(a)$-improvement

- Add irrelevant ops. to $\mathcal{L}_{\rm latt}$ which cancel $\mathcal{L}_1$-contributions

$$\mathcal{L}_{\rm latt} \to \mathcal{L}_{\rm latt} + ac_{\rm sw}(g_0^2)\overline{\psi}\sigma_{\mu\nu}F^{\rm latt}_{\mu\nu}\psi$$

$$m_{\rm q} \to m_{\rm q}(1 + b_{\rm m}(g_0^2)am_{\rm q}) \qquad g_0^2 \to g_0^2(1 + b_{\rm g}(g_0^2)am_{\rm q})$$

- $\mathcal{O}_1$ and $\mathcal{O}_2$ effects removed, but residual ${\rm O}(g_0^6 aM)$-effects from $\mathcal{O}_3$

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Large-mass continuum limit

Symanzik eff. action

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$$\mathcal{L}_{\mathrm{latt}} \approx \mathcal{L}_{\mathrm{QCD}} + \cancel{a\mathcal{L}_1} + a^2\mathcal{L}_2 + \dots \qquad \mathcal{L}_2 = \sum_{i=1}^{18} b_i(g^2)\mathcal{B}_i$$

$\mathrm{O}(a^2)$ contribution

$$\Delta(a) \equiv \alpha^{(3)}_{\mathrm{GF}}(\mu, M, a) - \alpha^{(3)}_{\mathrm{GF}}(\mu, M, 0)$$

Large-mass expansion

$[\, \mu \ll M \ll a^{-1} \,]$

$$\mathcal{L}_{\mathrm{QCD}} \approx \mathcal{L}_{\mathrm{YM}} + \frac{1}{M^2}\mathcal{L}_{2,\mathrm{dec}} + \dots$$

$$\mathcal{B}_i \approx M^2 d_{i0}\mathcal{D}_0 + \sum_{j=1}^{2} d_{ij}\mathcal{D}_j + \dots \qquad \mathcal{D}_0 = \frac{1}{4g^2}F^a_{\mu\nu}F^a_{\mu\nu}$$

Conclusion

$$\Delta(a) = \mathrm{O}(a^2M^2) + \mathrm{O}(a^2\mu^2)$$

LO anomalous dim:

- $\hat{\gamma}^{\mathcal{B}}_{\mathrm{min}} = -1/9$ for $\mathrm{O}(a^2M^2)$ term (Husung et al. '22; Husung '23)
- Only partial info available for $\mathrm{O}(a^2\mu^2)$ term

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$\mathrm{O}(a^2)$ contribution

$$\Delta(a) \propto a^2 \sum_{i=1}^{18} [\alpha_{\overline{\rm MS}}^{(3)}(a^{-1})]^{\hat{\gamma}_i^{\mathcal{B}}} b_i(\alpha) \int \mathrm{d}^4x \, \langle \mathcal{O}_{\rm GF}\, \mathcal{B}_i^{\rm RGI}(x)\rangle_{\rm QCD}^{\rm conn} + \ldots$$

Large-mass expansion $[\, \mu \ll M \ll a^{-1} \,]$

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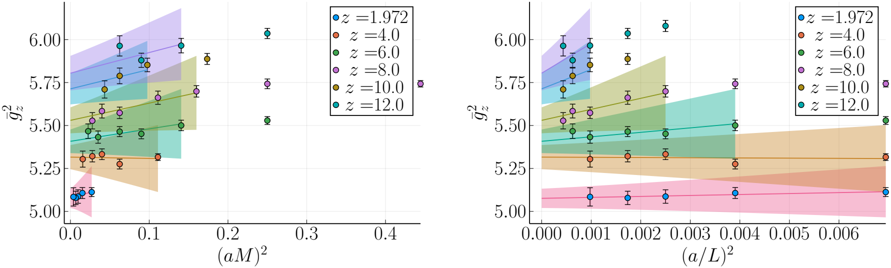
Continuum limit of the massive coupling



Global fit ansatz

$$\bar{g}_z^2 = C(z) + p_1[\alpha_{\overline{\rm MS}}^{(3)}(a^{-1})]^{\hat{\Gamma}}(a\mu_{\rm dec})^2 + p_2[\alpha_{\overline{\rm MS}}^{(3)}(a^{-1})]^{\hat{\Gamma}'}(aM)^2 \pm {\rm O}(aM)$$

$$\bar{g}_z^2/(4\pi) = \alpha_{\rm GF}^{(3)}(\mu_{\rm dec}, M, a) \qquad z = M/\mu_{\rm dec}$$

Remarks

- p_1, p_2 are z-independent; we find $p_1 \ll p_2$
- Γ, Γ', and aM varied to assess systematics
- Estimate of residual ${\rm O}(aM)$ effects using $\delta b_{\rm g} = b_{\rm g}^{\rm NLO}$
- Final results consider: $aM \leq 0.4$, $z \geq 4$, $\hat{\Gamma} = \hat{\Gamma}' = 0$

Large-mass extrapolation of $\Lambda^{(3)}_{\overline{\mathrm{MS}}}$

Pure-gauge running

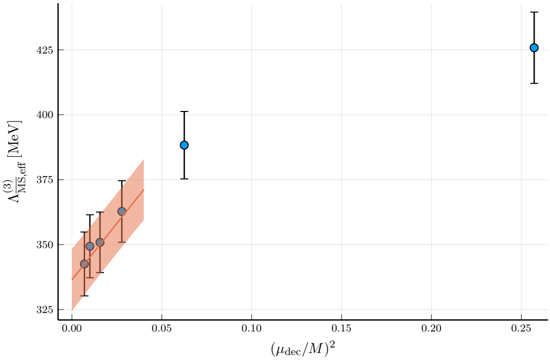
(MDB, Ramos '19)

$$\alpha^{(0)}_{\mathrm{GF}}(\mu_{\mathrm{dec}}) \stackrel{\mathrm{def.}}{=} \alpha^{(3)}_{\mathrm{GF}}(\mu_{\mathrm{dec}}, M)$$

$$\Lambda^{(0)}_{\overline{\mathrm{MS}}}/\mu_{\mathrm{dec}} = (\Lambda^{(0)}_{\overline{\mathrm{MS}}}/\Lambda^{(0)}_{\mathrm{GF}})\varphi^{(0)}_{\mathrm{GF}}(\alpha^{(0)}_{\mathrm{GF}}(\mu_{\mathrm{dec}}))$$

$$\alpha^{(3)}_{\mathrm{GF}}(\mu_{\mathrm{dec}}, 3.2\,\mathrm{GeV}) \;\Rightarrow\; \Lambda^{(0)}_{\overline{\mathrm{MS}}}/\mu_{\mathrm{dec}} = 0.719(16)$$

$$\alpha^{(3)}_{\mathrm{GF}}(\mu_{\mathrm{dec}}, 9.5\,\mathrm{GeV}) \;\Rightarrow\; \Lambda^{(0)}_{\overline{\mathrm{MS}}}/\mu_{\mathrm{dec}} = 0.797(21)$$



Master formula

$$\rho\, P^{(5\text{-loop})}_{0,3}(z/\rho) = \Lambda^{(0)}_{\overline{\mathrm{MS}}}/\mu_{\mathrm{dec}}$$

$$\rho = \Lambda^{(3)}_{\overline{\mathrm{MS}},\mathrm{eff}}/\mu_{\mathrm{dec}}$$

$$z = M/\mu_{\mathrm{dec}}$$

$$\mu_{\mathrm{dec}} = 789(15)\,\mathrm{MeV}$$

Fit ansatz

$$\Lambda^{(3)}_{\overline{\mathrm{MS}},\mathrm{eff}} = A + \frac{B}{z^2}\alpha_\star^{\hat{\Gamma}_m}$$

Result $[\; z \geq 6\,;\, \hat{\Gamma}_m = 0 \;]$

$$\Lambda^{(3)}_{\overline{\mathrm{MS}}} = 336(10)(6)_{aM}(3)_{\hat{\Gamma}_m}\,\mathrm{MeV}$$

The coupling from decoupling

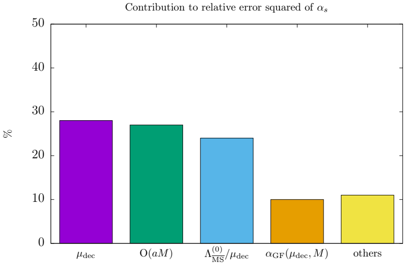
More decoupling

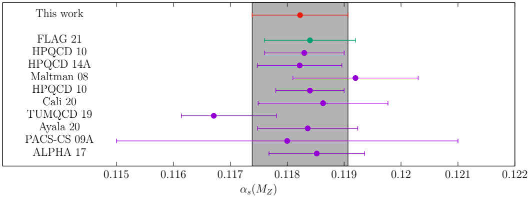
$$\Lambda^{(3)}_{\overline{\rm MS}} \xrightarrow{P^{(\text{5-loop})}_{3,4}(M_c/\Lambda^{(4)}_{\overline{\rm MS}})} \Lambda^{(4)}_{\overline{\rm MS}} \xrightarrow{P^{(\text{5-loop})}_{4,5}(M_b/\Lambda^{(5)}_{\overline{\rm MS}})} \Lambda^{(5)}_{\overline{\rm MS}} \xrightarrow{\beta^{(\text{5-loop})}_{\overline{\rm MS}}} \alpha^{(5)}_{\overline{\rm MS}}(m_Z)$$

Final result

$$\alpha^{(5)}_{\overline{\rm MS}}(m_Z) = 0.11823(69)(42)_{aM}(20)_{\hat\Gamma_m}(9)_{3\to5} = 0.1182(8)$$

FLAG 21: $\alpha^{(5)}_{\overline{\rm MS}}(m_Z) = 0.1184(8)$ **PDG 21:** $\alpha^{(5)}_{\overline{\rm MS}}(m_Z) = 0.1179(9)$ (FLAG '21; PDG '21)





Conclusions & Outlook

Conclusions

- Heavy-quark decoupling is a **powerful** tool for extracting α_s
- Allows us to replace the non-perturbative running from μ_{dec} to μ_{PT} in $N_{\mathrm{f}} = 3$ QCD with that in **pure Yang Mills**
- Current precision $\alpha_s(m_Z) \approx 0.7\%$ is comparable with the **most precise** lattice determinations
- Uncertainty is currently dominated by:
  1. Physical units of the scale μ_{dec}
  2. Residual $\mathrm{O}(aM)$ uncertainty
  3. Pure-gauge running

Outlook

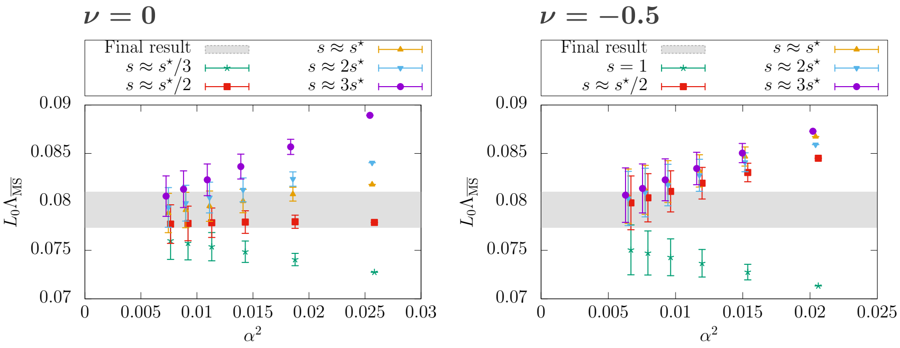
- **Short-term:** Reanalysis of α_s with no residual $\mathrm{O}(aM)$ uncertainty (coming soon)
- **Mid-term:** Compute $\Lambda^{(0)}_{\overline{\mathrm{MS}}}/\mu_{\mathrm{dec}}$ with $1/3$ of the uncertainty ($\approx 0.5\%$)
- **Long(er)-term:** More precise scale determination


Thanks

BACKUP

High-energy matching

(ALPHA Collab. '16, '18)



What was done

1. Match $\mathrm{SF}_\nu \to \overline{\mathrm{MS}}$ schemes at $\mu_n = 2^n\mu_0 = 2^n/L_0$ using

   $$\alpha_{\overline{\rm MS}}(s\mu_n) = \alpha_\nu(\mu_n) + c_1^\nu(s)\alpha_\nu^2(\mu_n) + c_2^\nu(s)\alpha_\nu^3(\mu_n) \qquad c_1^\nu(s^\star) = 0 \qquad |c_2^\nu(s^\star)| \lesssim 1$$

2. Extract $\Lambda_{\overline{\rm MS}}/\mu_0$ from $\alpha_{\overline{\rm MS}}(s\mu_n)$ using 5-loop $\beta_{\overline{\rm MS}}$-function
3. Assess size of PT truncation errors (of $\mathrm{O}(\alpha^2)$ in $\Lambda_{\overline{\rm MS}}/\mu_0$) through s-parameter dependence around $s^\star$

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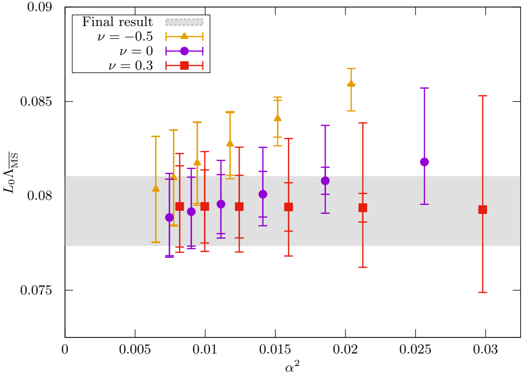

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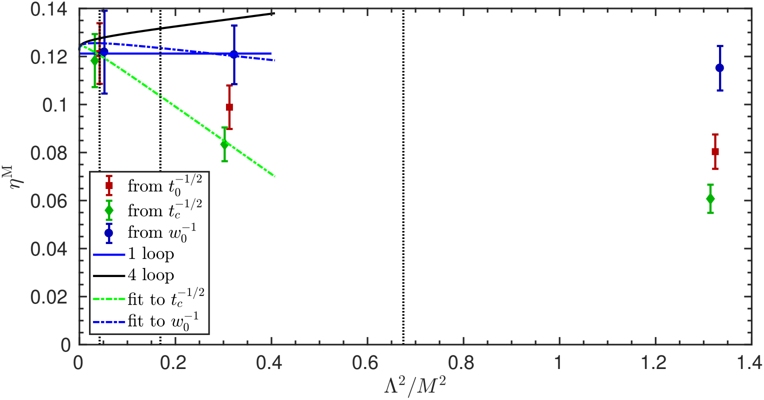
3. Assess size of PT truncation errors (of $\mathrm{O}(\alpha^2)$ in $\Lambda_{\overline{\rm MS}}/\mu_0$) through s-parameter dependence around $s^\star$

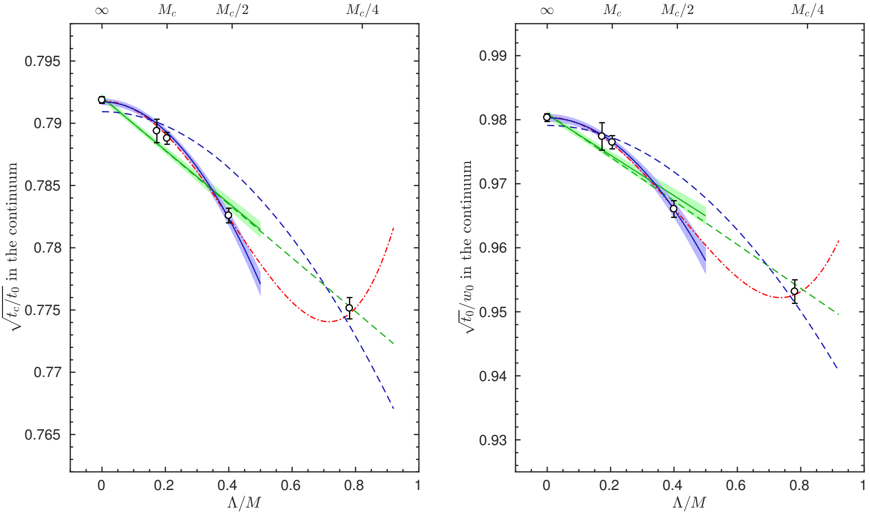
Non-perturbative decoupling tests



$$\eta^M = \frac{\partial \ln m_{\rm had}^{(N_{\rm f})}(M)}{\partial \ln M} = \frac{\partial \ln P_{0,2}^{\rm had}(M)}{\partial \ln M} \qquad m_{\rm had} = 1/\sqrt{t_0}, 1/\sqrt{t_c}, 1/w_0$$

(Athenodorou et al. '18)

Non-perturbative decoupling tests



(Knechtli et al. '17)

Pure Yang-Mills running

(MDB, Ramos '19)