# Non-planar 2-loop master integrals for Higgs + jet production

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Based on work in collaboration with:

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[arXiv: 1907.13156]

# Introduction

# Outline of the talk

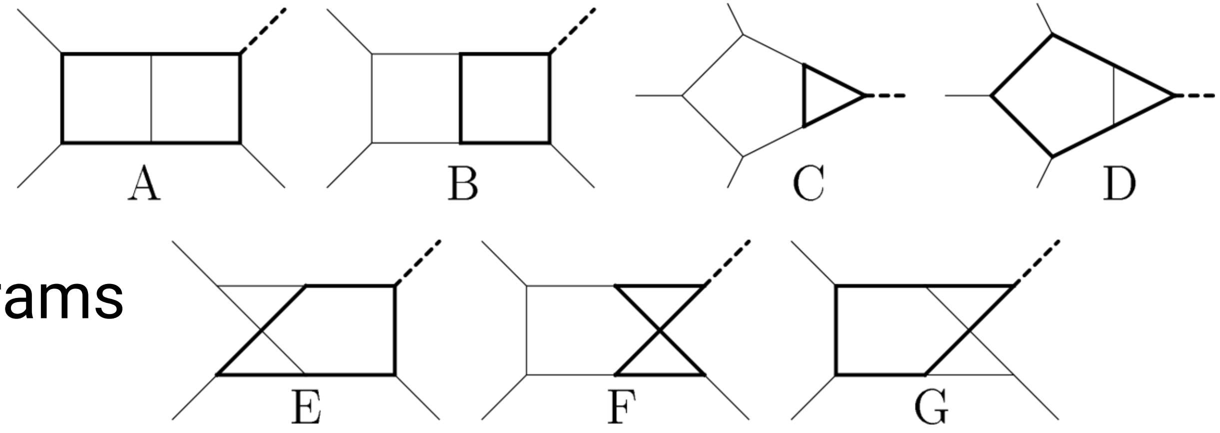
- Introduction and motivation
- Polylogarithmic sectors
  - Differential equations, canonical basis, analytic integration
- Derivation of boundary conditions
- Elliptic sectors
  - Series expansions
  - Results
- Conclusion

# Higgs + jet production

- Main production mode of the Higgs boson @ LHC is via gluon-gluon fusion
- Higgs does not couple directly to gluons
  - Interaction is mediated by a heavy quark loop, NLO @ 2-loop
- To this date, no NLO computation is available of the whole $p_T$-spectrum, including quark-mass effects for all quark flavors
  - An NLO computation including the top-quark mass but neglecting bottom-quark mass has been performed using sector decomposition for the integrals [Jones, Kerner, Luisoni, 2018]
  - Various computations have also been done in HEFT e.g. [Chen, Gehrmann, Glover, Jaquier, 2016]

# Higgs + jet production

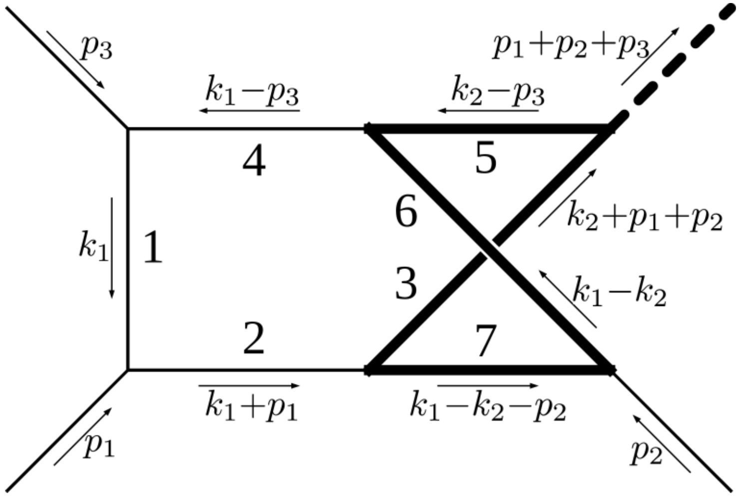
- Amplitude computation:
  - $\mathcal{O}(300)$ Feynman diagrams
  - Dirac algebra $\Rightarrow \mathcal{O}(20000)$ scalar diagrams
  - The diagrams fit into 7 topologies.
  - The planar families were computed in 2016.




[Bonciani et al, 1609.06685]

- We will focus on family F:

$$s = (p_1+p_2)^2, \quad t = (p_1+p_3)^2,$$
$$p_4^2 = (p_1+p_2+p_3)^2 = s+t+u.$$




$$P_1 = -k_1^2, \qquad P_4 = -(k_1-p_3)^2, \qquad P_7 = m^2 - (k_1-k_2-p_2)^2,$$
$$P_2 = -(k_1+p_1)^2, \qquad P_5 = m^2 - (k_2-p_3)^2, \qquad P_8 = m^2 - k_2^2,$$
$$P_3 = m^2 - (k_2+p_1+p_2)^2, \qquad P_6 = m^2 - (k_1-k_2)^2, \qquad P_9 = m^2 - (k_1-k_2-p_1-p_2)^2.$$

# Family F

## Master integrals

- IBP-reduction:
  - 73 master integrals
  - Default FIRE basis: $\mathcal{O}$(1 GB)
  - More suitable (pre-canonical) basis: $\mathcal{O}$(100 MB)
  - Possible using either FIRE or KIRA

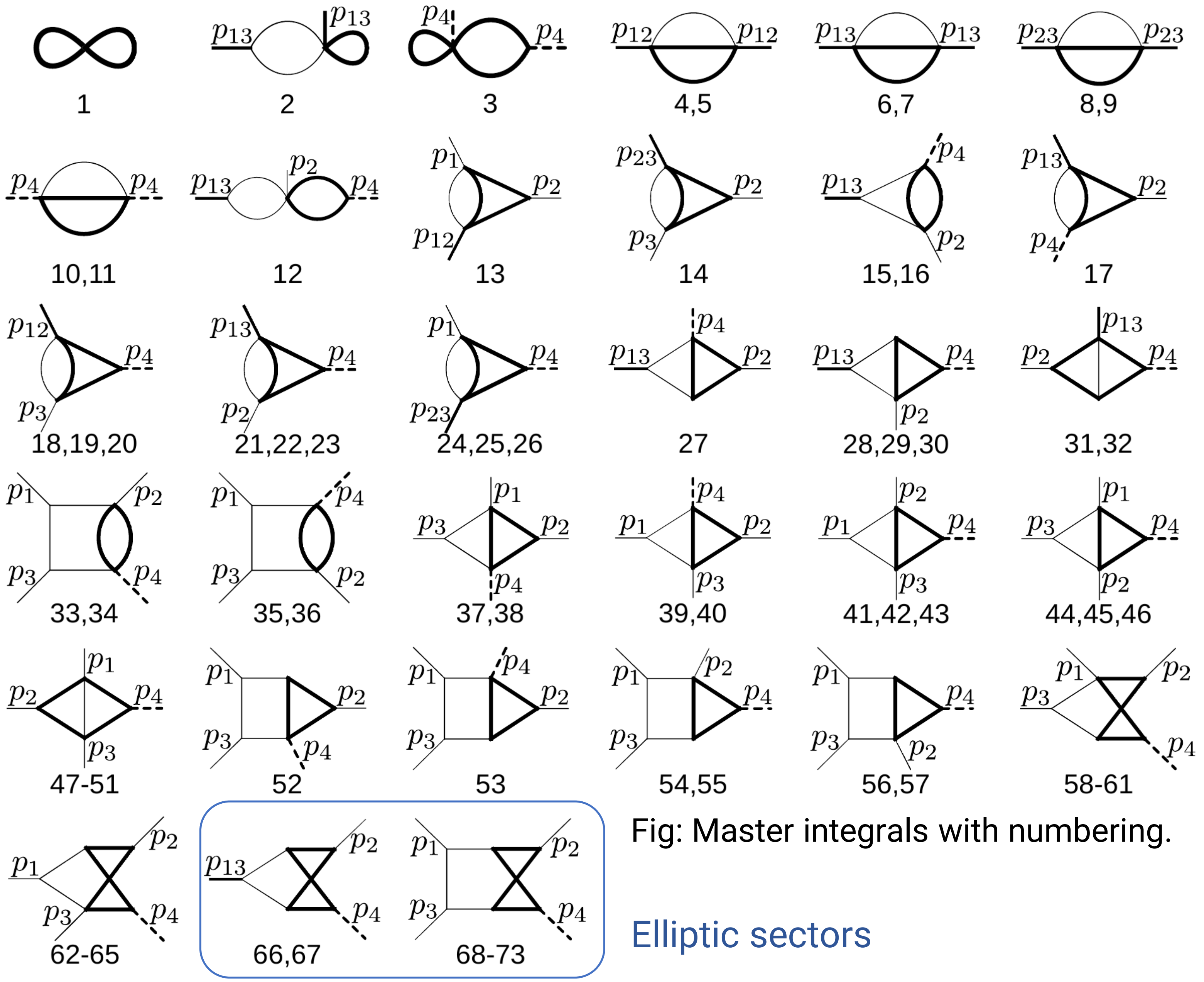



Fig: Master integrals with numbering.

Elliptic sectors

# Polylogarithmic sectors

# Canonical basis

- Master integrals satisfy closed systems of first order linear differential equations:

$$\frac{\partial}{\partial s_i}\vec{\mathrm{I}} = \mathbf{M}_{s_i}\vec{\mathrm{I}}\,,$$

[Kotikov, 1991], [Remiddi, 1997]
[Gehrmann, Remiddi, 2000]

- Change of basis: $\vec{B} = \mathbf{T}\vec{\mathrm{I}} \;\Rightarrow\; \frac{\partial}{\partial s_i}\vec{B} = \left[\left(\partial_{s_i}\mathbf{T}\right)\mathbf{T}^{-1} + \mathbf{T}\mathbf{M}_{s_i}\mathbf{T}^{-1}\right]\vec{B}\,.$

- Canonical basis conjecture $\exists\mathbf{T}$ :

[Henn, 2013]

$$\left(\partial_{s_i}\mathbf{T}\right)\mathbf{T}^{-1} + \mathbf{T}\mathbf{M}_{s_i}\mathbf{T}^{-1} = \epsilon\mathbf{A}_{s_i}\,, \qquad d\vec{B} = \epsilon d\tilde{\mathbf{A}}\vec{B}\,,$$

- Polylogarithmic case:

$$d\tilde{\mathbf{A}} = \sum_{i\in\mathcal{A}}\mathbf{A}_i d\log(l_i)$$

# Canonical basis

- Solutions given in terms of Chen's iterated integrals: [Chen, 1977]

$$d\vec{B} = \epsilon \left( d\tilde{\mathbf{A}} \right) \vec{B} \quad \Rightarrow \quad \vec{B} = \mathbb{P}\exp\left[ \epsilon \int_\gamma d\tilde{\mathbf{A}} \right] \vec{B}_{\text{boundary}}$$

$$\vec{B} = \vec{B}^{(0)}_{\text{boundary}} + \sum_{k\geq 1} \epsilon^k \sum_{j=1}^{k} \int_0^1 \gamma^*(d\tilde{\mathbf{A}})(t_1) \int_0^{t_1} \gamma^*(d\tilde{\mathbf{A}})(t_2) \dots \int_0^{t_{j-1}} \gamma^*(d\tilde{\mathbf{A}})(t_j)\, \vec{B}^{(k-j)}_{\text{boundary}}$$

- Which may in turn yield MPL's, iterated Eisenstein series, … depending on the kernels of $d\tilde{\mathbf{A}}$ [See works Adams & Weinzierl for the elliptic case]

- In the presence of non-simultaneously rationalizable square roots it is not manifest how to find solutions in terms of MPL's, even when $d\tilde{\mathbf{A}} = \sum_{i\in\mathcal{A}} \mathbf{A}_i d\log(l_i)$

# Finding the canonical basis

- Many publicly available algorithms (Epsilon, Fuchsia, Canonica, ..) [Lee, 1411.0911] [Prausa, 1701.00725] [Gituliar, Magerya, 1701.04269] [Meyer, 1705.06252]
- Canonical basis can also be computed "manually"
  - Not constrained by black-box program
- (Outline of) strategy:
  - Work sector by sector on the maximal cut
  - "Pre-canonical" basis $\partial_{s_i} \vec{B}^{\text{sector,cut}}_{\text{precan.}} = \left(A_{s_i,1} + \epsilon A_{s_i,2}\right) \vec{B}^{\text{sector,cut}}_{\text{precan.}}\,,$
  - Change of basis $\vec{B}^{\text{sector,cut}} = \mathbf{T}\vec{B}^{\text{sector,cut}}_{\text{precan.}}$
  - $\Rightarrow \dfrac{\partial}{\partial s_i}\vec{B}^{\text{sector,cut}} = \left[(\partial_{s_i}\mathbf{T})\,\mathbf{T}^{-1} + \mathbf{T}\left(A_{s_i,1} + \epsilon A_{s_i,2}\right)\mathbf{T}^{-1}\right]\vec{B}^{\text{sector,cut}}\,.$
  - Solve for: $(\partial_{s_i}\mathbf{T}) + \mathbf{T}A_{s_i,1} = 0$

# Finding the canonical basis

- After the previous procedure, the homogeneous blocks are in canonical form. By systematically shifting out terms from sub-topologies, we may turn the full set of differential equations canonical. [Gehrmann, von Manteuffel, Tancredi, Weihs, 1404.4853]
- For example, suppose that the $(i, j)$-th entry of the matrix has the form $A + \epsilon\, B$. One may shift, $B_i \to B_i + \alpha(\ldots)B_j$, where $\alpha$ depends on the external scales.
- This returns a differential equation for $\alpha(\ldots)$, which may be solved to put the $\epsilon^0$ term to zero.
- After all basis shifts, we are left with a system for each $s_i$: $\partial_{s_i}\vec{B} = \epsilon \mathbf{A}_{s_i}\vec{B}$ .

# Finding the canonical basis

$$B_{58} = \epsilon^4 r_5 r_{15} \mathrm{I}_{1,0,1,1,1,1,1,0,0}\,,$$

$$B_{59} = \epsilon^4 \left(p_4^2 - t\right) \left(\mathrm{I}_{1,0,1,1,0,1,1,0,0} - \mathrm{I}_{1,0,1,1,1,1,1,-1,0}\right)\,,$$

$$B_{60} = \frac{s^2 - p_4^2 s + t^2 - t p_4^2}{p_4^2 - s} \mathrm{I}_{1,0,1,0,1,1,1,0,0}\epsilon^4 + \left(-p_4^2 + s + t\right)\left(\mathrm{I}_{1,-1,1,1,1,1,1,0,0} + t\mathrm{I}_{1,0,1,1,1,1,1,0,0}\right)\epsilon^4 + \frac{t}{p_4^2 - s}\Bigg(\frac{1}{4}(B_6 + B_{10}) + \frac{1}{2}(B_8 - B_{13} - B_{14} + B_{18} + B_{21}) - B_{22} - B_{44} + B_{46} + B_{50} - B_{59}\Bigg)\,,$$

$$B_{61} = \epsilon^3 r_1 r_6 \left(\left(-s - t + p_4^2\right)\left((-2\epsilon)\mathrm{I}_{1,0,1,1,1,1,1,0,0} - \mathrm{I}_{1,0,1,1,1,0,2,0,0}\right) + s\mathrm{I}_{1,0,2,1,0,1,1,0,0} + \left(t - p_4^2\right)\mathrm{I}_{1,0,2,1,1,1,1,-1,0}\right)\,,$$

$$B_{62} = \epsilon^4 r_2 r_{14} \mathrm{I}_{1,1,1,0,1,1,1,0,0}\,,$$

$$B_{63} = \epsilon^4 \left(p_4^2 - t\right)\left(\mathrm{I}_{1,1,1,0,1,1,0,0,0} - \mathrm{I}_{1,1,1,0,1,1,1,0,-1}\right)\,,$$

$$B_{64} = s\mathrm{I}_{1,1,1,-1,1,1,1,0,0}\epsilon^4 + (st)\mathrm{I}_{1,1,1,0,1,1,1,0,0}\epsilon^4 + \frac{t}{s+t}\Bigg(\frac{1}{4}(-B_6 - B_{10}) + B_{22} + \frac{1}{2}(-B_4 + B_{13} + B_{14} - B_{21} - B_{24}) - B_{31} + B_{41} - B_{43} - B_{50}\Bigg) + \frac{1}{s+t}\left(\left(-s^2 - ts - 2t^2 + 2tp_4^2\right)\mathrm{I}_{1,0,1,0,1,1,1,0,0}\epsilon^4 + sB_{63}\right)\,,$$

$$r_1 = \sqrt{-p_4^2}\,, \quad r_2 = \sqrt{-s}\,,$$
$$r_3 = \sqrt{-t}\,, \quad r_4 = \sqrt{t - p_4^2}\,,$$
$$r_5 = \sqrt{s + t - p_4^2}\,, \quad r_6 = \sqrt{4m^2 - p_4^2}\,,$$
$$r_7 = \sqrt{4m^2 - s}\,, \quad r_8 = \sqrt{4m^2 - t}\,,$$
$$r_9 = \sqrt{4m^2 - p_4^2 + t}\,, \quad r_{10} = \sqrt{4m^2 - p_4^2 + s + t}\,,$$
$$r_{11} = \sqrt{4m^2(p_4^2 - s - t) + st}\,, \quad r_{12} = \sqrt{4m^2 t + s(p_4^2 - s - t)}\,,$$
$$r_{13} = \sqrt{4m^2 s + t(p_4^2 - s - t)}\,, \quad r_{14} = \sqrt{4m^2 t(s + t - p_4^2) - (p_4^2)^2 s}\,,$$
$$r_{15} = \sqrt{-4m^2 st + (p_4^2)^2 (s + t - p_4^2)}\,, \quad r_{16} = \sqrt{16m^2 t + (p_4^2 - t)^2}\,.$$

# Finding the canonical basis

- It remains to find the matrix $\tilde{\mathbf{A}}$ such that $\partial_{s_i}\tilde{\mathbf{A}} = \mathbf{A}_{s_i}$
- The matrix can be computed using:

$$\tilde{\mathbf{A}}_1 := \int \mathbf{A}_{s_1} ds_1 \,,$$

$$\tilde{\mathbf{A}}_i := \int \left( \mathbf{A}_{s_i} - \partial_{s_i} \sum_{j=1}^{i-1} \mathbf{A}_j \right) ds_i \,, \qquad i = 2, \ldots, 4 \,. \qquad \tilde{\mathbf{A}} = \sum_i \tilde{\mathbf{A}}_i$$

- $\tilde{\mathbf{A}}_i$ should not depend on the variables $s_j$, with $j < i$
- The integrations may be hard. A trial-and-error approach sufficed:
  - Plug in numbers • Try different integration orders • Rationalize square roots

# Deriving the alphabet

- The previous integration formula for $\tilde{\mathbf{A}}$ yields complicated expressions
- We seek a linearly independent alphabet $\mathcal{A}$ for $\tilde{\mathbf{A}}$, such that every matrix entry satisfies: $\tilde{\mathbf{A}}_{ij} = \sum_k a_{ijk} \log(l_i)$, for $a_{ijk} \in \mathbb{Q}$ and $l_i \in \mathcal{A}$
- In addition we want the alphabet to be minimal:

$$\mathrm{Span}_{\mathbb{Q}}\{\tilde{\mathbf{A}}_{ij}\} = \mathrm{Span}_{\mathbb{Q}}\mathcal{A}$$

- (in the notation: $\mathrm{Span}_{\mathbb{Q}}\mathcal{A}^{\mathrm{example}} = \mathrm{Span}_{\mathbb{Q}}\{\log a, \log b\}$ for $\mathcal{A}^{\mathrm{example}} = \{a, b\}$)

# Deriving the alphabet

$$2\,\mathrm{Log}\left[\sqrt{4\,mm-pp4}\,\sqrt{-s}\,\sqrt{-t}-\sqrt{-pp4}\,\sqrt{4\,mm\,(pp4-s-t)+s\,t}\right]-2\,\mathrm{Log}\left[\sqrt{4\,mm-pp4}\,\sqrt{-s}\,\sqrt{-t}+\sqrt{-pp4}\,\sqrt{4\,mm\,(pp4-s-t)+s\,t}\right]-$$
$$\mathrm{Log}\left[\sqrt{4\,mm-pp4}\,\sqrt{-pp4}\,\sqrt{-s}+pp4\sqrt{-s}-\sqrt{-s}\,t-\sqrt{-t}\,\sqrt{4\,mm\,(pp4-s-t)+s\,t}\right]-\mathrm{Log}\left[\sqrt{4\,mm-pp4}\,\sqrt{-pp4}\,\sqrt{-s}-pp4\sqrt{-s}+\sqrt{-s}\,t-\sqrt{-t}\,\sqrt{4\,mm\,(pp4-s-t)+s\,t}\right]+$$
$$\mathrm{Log}\left[\sqrt{4\,mm-pp4}\,\sqrt{-pp4}\,\sqrt{-s}+pp4\sqrt{-s}-\sqrt{-s}\,t+\sqrt{-t}\,\sqrt{4\,mm\,(pp4-s-t)+s\,t}\right]+\mathrm{Log}\left[\sqrt{4\,mm-pp4}\,\sqrt{-pp4}\,\sqrt{-s}-pp4\sqrt{-s}+\sqrt{-s}\,t+\sqrt{-t}\,\sqrt{4\,mm\,(pp4-s-t)+s\,t}\right]$$

$$=$$

$$l_{57}=\mathrm{Log}\left[\frac{-2\,mm\,pp4\,s+2\,mm\,s^2+2\,mm\,pp4\,t+2\,mm\,s\,t-pp4\,s\,t-\sqrt{4\,mm-pp4}\,\sqrt{-pp4}\,\sqrt{-s}\,\sqrt{-t}\,\sqrt{4\,mm\,pp4-4\,mm\,s-4\,mm\,t+s\,t}}{-2\,mm\,pp4\,s+2\,mm\,s^2+2\,mm\,pp4\,t+2\,mm\,s\,t-pp4\,s\,t+\sqrt{4\,mm-pp4}\,\sqrt{-pp4}\,\sqrt{-s}\,\sqrt{-t}\,\sqrt{4\,mm\,pp4-4\,mm\,s-4\,mm\,t+s\,t}}\right]\pmod{i\pi}$$

# Testing independence of letters

- Consider a linearly independent alphabet $\mathcal{A}$ with $n$ letters
- We seek to test if $a \in \mathrm{Span}_{\mathbb{Q}}\mathcal{A}$, for some letter $a$
- Let $\mathcal{A}^+ = \mathcal{A} \cup \{a\}$, and let $S$ denote a large set of distinct numerical samples of the external scales $S = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$
- Set up a matrix $L$ whose rows are derivatives of $\mathcal{A}^+$ evaluated in the samples of $S$

$$\begin{pmatrix}
(\partial_{s_1}\log(l_1))(\vec{x_1}) & \cdots & (\partial_{s_1}\log(l_n))(\vec{x_1}) & (\partial_{s_1}\log(a))(\vec{x_1}) \\
(\partial_{s_2}\log(l_1))(\vec{x_1}) & \cdots & (\partial_{s_2}\log(l_n))(\vec{x_1}) & (\partial_{s_2}\log(a))(\vec{x_1}) \\
\vdots & & \vdots & \\
(\partial_{s_k}\log(l_1))(\vec{x_1}) & \cdots & (\partial_{s_k}\log(l_n))(\vec{x_1}) & (\partial_{s_k}\log(a))(\vec{x_1}) \\
(\partial_{s_1}\log(l_1))(\vec{x_2}) & \cdots & (\partial_{s_1}\log(l_n))(\vec{x_2}) & (\partial_{s_1}\log(a))(\vec{x_2}) \\
(\partial_{s_2}\log(l_1))(\vec{x_2}) & \cdots & (\partial_{s_2}\log(l_n))(\vec{x_2}) & (\partial_{s_2}\log(a))(\vec{x_2}) \\
\vdots & \vdots & \vdots &
\end{pmatrix}$$

# Testing independence of letters

- Then: $a$ is independent of $\mathcal{A}$ if $\mathrm{rank}(L) = n + 1$
- Similarly we find factorizations over the alphabet
- Suppose now that $L$ is a $n \times n$ full-rank matrix of samples of derivatives of $\mathcal{A}$ . Suppose furthermore that: $\log(a) = \sum_i c_i \log(l_i)$.
- Compute the row vector $\vec{v} = \{\partial_{s_i} \log(a(\vec{x}_j))\}$ where the derivatives and samples are computed in the same order as for $L$
- Then note $L.\vec{c} = \vec{v} \Rightarrow \vec{c} = L^{-1}\vec{v}$

# Testing independence of letters

- If $\log(a) \neq \sum_i c_i \log(l_i)$, for $c_i \in \mathbb{Q}$, then the resulting vector $\vec{c}$ depends on the samples that were taken for $L$
- Hence, setup 2 sampling matrices $L_1, L_2$, and compute:

$$\vec{\tilde{c}} = L_1^{-1}\overrightarrow{v_1} - L_2^{-1}\overrightarrow{v_2}$$

- If $\vec{\tilde{c}} \neq 0$, then we conclude $a$ doesn't factorize over $\mathcal{A}$. Otherwise we have found the factorization.
- Note: this is a fast procedure, only requiring the computation of $2n$ derivatives, and 2 matrix-vector multiplications

# Deriving the alphabet

- We use the previous methods to obtain an independent set of letters for $\tilde{\mathbf{A}}$
- Let $\mathcal{A}^{\mathrm{oc}}$ be a set of irreducible arguments of logarithms of in $\tilde{\mathbf{A}}$
- Order $\mathcal{A}^{\mathrm{oc}}$ by complexity (e.g. using Bytecount[...])
- Accumulate linearly independent elements in $\mathcal{A}^{\mathrm{idp}}$ until

$$\mathrm{Span}_{\mathbb{Q}}\mathcal{A}^{\mathrm{oc}} = \mathrm{Span}_{\mathbb{Q}}\mathcal{A}^{\mathrm{idp}}$$

# Deriving the alphabet

- Next, we make the letters manifestly symmetric:
- Case 1: $$\log\left(a \pm b\sqrt{c}\right) = \frac{1}{2}\left[\pm \log\left(\frac{a/b+\sqrt{c}}{a/b-\sqrt{c}}\right) + \log\left(a^2 - b^2 c\right)\right] \pmod{i\pi}\,.$$
- Case 2: $$\begin{aligned}\log\left(a + b\sqrt{c} + d\sqrt{e}\right) &= \frac{1}{4}\log\left(\frac{a^2+b^2c-d^2e+2ab\sqrt{c}}{a^2+b^2c-d^2e-2ab\sqrt{c}}\right)\\ &-\frac{1}{4}\log\left(\frac{a^2-b^2c-d^2e+2bd\sqrt{c}\sqrt{e}}{a^2-b^2c-d^2e-2bd\sqrt{c}\sqrt{e}}\right)\\ &+\frac{1}{4}\log\left(\frac{a^2-b^2c+d^2e+2ad\sqrt{e}}{a^2-b^2c+d^2e-2ad\sqrt{e}}\right)\\ &+\frac{1}{4}\log\left(a^4-2a^2\left(b^2c+d^2e\right)+\left(b^2c-d^2e\right)^2\right) \pmod{i\pi}\,.\end{aligned}$$
- In general: multiply by all combinations of conjugates, and recurse

# Deriving the alphabet

- Lastly, we minimize the alphabet.
- Factor each $\tilde{\mathbf{A}}_{ij}$ over $\mathcal{A}^{\mathrm{sym}}$, and write the entries out in vector notation. (i.e.: $\log(l_1) + 2\log(l_2) + 0.\log(l_3) \to (1,2,0)$)
- Compute the matrix rank to get the number of independent entries (69 for family F). If it is lower than $\#\mathcal{A}^{\mathrm{sym}}$, some letters can be combined.

# Deriving the polylogarithmic alphabet

- We read off reoccurring combinations as follows:
- Sort $\{\tilde{\mathbf{A}}_{ij}\}$ by size (using Bytecount[..])
- Pick a maximal subset of linearly independent entries, starting from the simplest ones.
- Read off which letters appear only in combinations, and combine:

$$l_{63} = \frac{\left(\frac{q_3}{2} - r_3 r_{11} r_{14}\right)\left(\frac{q_{17}}{p_4^2} - r_3 r_{11} r_{14}\right)}{\left(\frac{q_3}{2} + r_3 r_{11} r_{14}\right)\left(\frac{q_{17}}{p_4^2} + r_3 r_{11} r_{14}\right)} \quad l_{64} = \frac{\left(-\frac{q_5}{p_4^2} + r_2 r_{12} r_{15}\right)\left(-\frac{q_9}{2} + r_2 r_{12} r_{15}\right)}{\left(-\frac{q_5}{p_4^2} - r_2 r_{12} r_{15}\right)\left(-\frac{q_9}{2} - r_2 r_{12} r_{15}\right)},$$

# Alphabet

$$l_1 = m^2\,, \qquad l_2 = p_4^2\,,$$
$$l_3 = s\,, \qquad l_4 = t\,,$$
$$l_5 = s+t\,, \qquad l_6 = -4m^2 + p_4^2\,,$$
$$l_7 = -s + p_4^2\,, \qquad l_8 = -t + p_4^2\,,$$
$$l_9 = s + t - p_4^2\,, \qquad l_{10} = 4m^2 - s\,,$$
$$l_{11} = 4m^2 - t\,, \qquad l_{12} = 4m^2 + t - p_4^2\,,$$
$$l_{13} = 4m^2 + s + t - p_4^2\,, \qquad l_{14} = s^2 + m^2 p_4^2 - s p_4^2\,,$$
$$l_{15} = t^2 + m^2 p_4^2 - t p_4^2\,, \qquad l_{16} = -4m^2 s - 4m^2 t + st + 4m^2 p_4^2\,,$$
$$l_{17} = -s^2 + 4m^2 t - st + s p_4^2\,, \qquad l_{18} = 4m^2 s - st - t^2 + t p_4^2\,,$$
$$l_{19} = m^2 s^2 + 2m^2 st + m^2 t^2 - st p_4^2\,, \qquad l_{20} = s^2 + 2st + t^2 + m^2 p_4^2 - s p_4^2 - t p_4^2\,,$$
$$l_{21} = -4m^2 st - 4m^2 t^2 + 4m^2 t p_4^2 + s p_4^4\,, \qquad l_{22} = 4m^2 st - s p_4^4 - t p_4^4 + p_4^6\,,$$
$$l_{23} = q_{11}\,, \qquad l_{24} = q_{12}\,,$$
$$l_{25} = \frac{1 + \frac{r_7}{r_2}}{-1 + \frac{r_7}{r_2}}\,, \qquad l_{26} = \frac{1 + \frac{r_8}{r_3}}{-1 + \frac{r_8}{r_3}}\,,$$
$$l_{27} = \frac{-p_4^2 + r_1 r_6}{-p_4^2 - r_1 r_6}\,, \qquad l_{28} = \frac{1 + \frac{r_5}{r_{10}}}{-1 + \frac{r_5}{r_{10}}}\,,$$
$$l_{29} = \frac{(t - p_4^2) + r_4 r_9}{(t - p_4^2) - r_4 r_9}\,, \qquad l_{30} = \frac{(2s - p_4^2) - r_1 r_6}{(2s - p_4^2) + r_1 r_6}\,,$$
$$l_{31} = \frac{(2s + 2t - p_4^2) - r_1 r_6}{(2s + 2t - p_4^2) + r_1 r_6}\,, \qquad l_{32} = \frac{-\frac{2m^2 - t}{t} - \frac{r_6}{r_1}}{-\frac{2m^2 - t}{t} + \frac{r_6}{r_1}}\,,$$
$$l_{33} = \frac{1 + \frac{r_{11}}{r_2 r_3}}{1 - \frac{r_{11}}{r_2 r_3}}\,, \qquad l_{34} = \frac{\frac{q_1}{t} - r_1 r_6}{\frac{q_1}{t} + r_1 r_6}\,,$$
$$l_{35} = \frac{\frac{1}{p_4^2} + \frac{r_2}{r_{14}}}{-\frac{1}{p_4^2} + \frac{r_2}{r_{14}}}\,, \qquad l_{36} = \frac{-\frac{2m^2 s + 2m^2 t - t p_4^2}{t} - r_1 r_6}{-\frac{2m^2 s + 2m^2 t - t p_4^2}{t} + r_1 r_6}\,,$$

....

....

$$l_{51} = \frac{t - p_4^2 \qquad r_2 r_5}{\frac{t + p_4^2}{t - p_4^2} + \frac{r_{12}}{r_2 r_5}}\,, \qquad l_{52} = \frac{s - p_4^2 \qquad r_3 r_5}{\frac{s + p_4^2}{s - p_4^2} + \frac{r_{13}}{r_3 r_5}}\,,$$
$$l_{53} = \frac{1 + \frac{r_1 r_{11}}{r_2 r_3 r_6}}{1 - \frac{r_1 r_{11}}{r_2 r_3 r_6}}\,, \qquad l_{54} = \frac{q_2 - r_1 r_4 r_6 r_9}{q_2 + r_1 r_4 r_6 r_9}\,,$$
$$l_{55} = \frac{(2m^2 t - 2m^2 s + st + t^2 - t p_4^2) + r_3 r_{10} r_{13}}{(2m^2 t - 2m^2 s + st + t^2 - t p_4^2) - r_3 r_{10} r_{13}}\,, \qquad l_{56} = \frac{q_4 - r_5 r_8 r_{13}}{q_4 + r_5 r_8 r_{13}}\,,$$
$$l_{57} = \frac{q_8 - r_1 r_2 r_3 r_6 r_{11}}{q_8 + r_1 r_2 r_3 r_6 r_{11}}\,, \qquad l_{58} = \frac{-q_7 - r_1 r_2 r_6 r_{14}}{-q_7 + r_1 r_2 r_6 r_{14}}\,,$$
$$l_{59} = \frac{q_{13} - r_1 r_5 r_6 r_{15}}{q_{13} + r_1 r_5 r_6 r_{15}}\,, \qquad l_{60} = \frac{-q_{15} + r_1 r_3 r_5 r_6 r_{13}}{-q_{15} - r_1 r_3 r_5 r_6 r_{13}}\,,$$
$$l_{61} = \frac{q_{14} - r_1 r_2 r_5 r_6 r_{12}}{q_{14} + r_1 r_2 r_5 r_6 r_{12}}\,, \qquad l_{62} = \frac{q_{16} - r_1 r_3 r_5 r_6 r_{13}}{q_{16} + r_1 r_3 r_5 r_6 r_{13}}\,,$$
$$l_{63} = \frac{\left(\frac{q_3}{2} - r_3 r_{11} r_{14}\right)\left(\frac{q_{17}}{p_4^2} - r_3 r_{11} r_{14}\right)}{\left(\frac{q_3}{2} + r_3 r_{11} r_{14}\right)\left(\frac{q_{17}}{p_4^2} + r_3 r_{11} r_{14}\right)}\,, \qquad l_{64} = \frac{\left(-\frac{q_5}{p_4^2} + r_2 r_{12} r_{15}\right)\left(-\frac{q_9}{2} + r_2 r_{12} r_{15}\right)}{\left(-\frac{q_5}{p_4^2} - r_2 r_{12} r_{15}\right)\left(-\frac{q_9}{2} - r_2 r_{12} r_{15}\right)}\,,$$
$$l_{65} = \frac{\left(-\frac{q_6}{p_4^2} + r_3 r_{13} r_{15}\right)\left(-\frac{q_{10}}{2} + r_3 r_{13} r_{15}\right)}{\left(-\frac{q_6}{p_4^2} - r_3 r_{13} r_{15}\right)\left(-\frac{q_{10}}{2} - r_3 r_{13} r_{15}\right)}\,, \qquad l_{67} = \frac{\left(\frac{q_{18}}{2} + r_5 r_{12} r_{14}\right)\left(-\frac{q_{19}}{p_4^2} + r_5 r_{12} r_{14}\right)}{\left(\frac{q_{18}}{2} - r_5 r_{12} r_{14}\right)\left(-\frac{q_{19}}{p_4^2} - r_5 r_{12} r_{14}\right)}\,,$$
$$l_{66} = \frac{\frac{q_{24}}{(2s + t - p_4^2)(t + p_4^2)} + r_1 r_2 r_5 r_6 r_{12}}{\frac{q_{24}}{(2s + t - p_4^2)(t + p_4^2)} - r_1 r_2 r_5 r_6 r_{12}}\,, \qquad l_{68} = \frac{\left(-\frac{q_{22}}{p_4^2} - r_2 r_3 r_5 r_{11} r_{15}\right)\left(\frac{q_{20}}{2} + r_2 r_3 r_5 r_{11} r_{15}\right)}{\left(\frac{q_{20}}{2} - r_2 r_3 r_5 r_{11} r_{15}\right)\left(-\frac{q_{22}}{p_4^2} + r_2 r_3 r_5 r_{11} r_{15}\right)}\,,$$
$$l_{69} = \frac{\left(-\frac{q_{23}}{p_4^2} - r_2 r_3 r_5 r_{13} r_{14}\right)\left(\frac{q_{21}}{2} + r_2 r_3 r_5 r_{13} r_{14}\right)}{\left(\frac{q_{21}}{2} - r_2 r_3 r_5 r_{13} r_{14}\right)\left(-\frac{q_{23}}{p_4^2} + r_2 r_3 r_5 r_{13} r_{14}\right)}\,.$$

# Alphabet

- 69 letters in total
- We labeled the following roots:

$$r_1 = \sqrt{-p_4^2}\,, \quad r_2 = \sqrt{-s}\,,$$
$$r_3 = \sqrt{-t}\,, \quad r_4 = \sqrt{t-p_4^2}\,,$$
$$r_5 = \sqrt{s+t-p_4^2}\,, \quad r_6 = \sqrt{4m^2-p_4^2}\,,$$
$$r_7 = \sqrt{4m^2-s}\,, \quad r_8 = \sqrt{4m^2-t}\,,$$
$$r_9 = \sqrt{4m^2-p_4^2+t}\,, \quad r_{10} = \sqrt{4m^2-p_4^2+s+t}\,,$$
$$r_{11} = \sqrt{4m^2(p_4^2-s-t)+st}\,, \quad r_{12} = \sqrt{4m^2t+s(p_4^2-s-t)}\,,$$
$$r_{13} = \sqrt{4m^2s+t(p_4^2-s-t)}\,, \quad r_{14} = \sqrt{4m^2t(s+t-p_4^2)-\left(p_4^2\right)^2 s}\,,$$
$$r_{15} = \sqrt{-4m^2st+\left(p_4^2\right)^2(s+t-p_4^2)}\,, \quad r_{16} = \sqrt{16m^2t+\left(p_4^2-t\right)^2}\,.$$

- The roots appear in 10 independent combinations:

$$\{r_1r_6, r_2r_7, r_3r_8, r_4r_9, r_5r_{10}, r_2r_3r_{11}, r_2r_5r_{12}, r_3r_5r_{13}, r_2r_{14}, r_5r_{15}\}\,.$$

- The roots are not simultaneously rationalizable

# Analytic integration

- The polylogarithmic sectors of family F contain 10 independent square roots, which may not be simultaneously rationalized.
- Thus, it is not manifest how to rewrite the Chen iterated integrals in terms of multiple polylogarithms.
- We may perform the integration at weight 2 in terms of logarithms and dilogarithms, by matching an ansatz with the symbol.
- Weight 3 and 4 may be obtained as 1-fold integrals over the weight 2 result.
- We illustrate the integration in a region where the canonical basis is real-valued.

# Analytic integration

- Consider the region: $\mathcal{R} : r_1 > 0, \ldots, r_{15} > 0 \cap \mathcal{E}$ , where $\mathcal{E}$ is the Euclidean region. After simplifying, we have:

$$\mathcal{R}: \quad t < -4m^2 \,\&\, s < -4m^2 \,\&\, \left( \left( s \leq t \,\&\, \frac{4m^2(s+t) - st}{4m^2} < p_4^2 < \frac{-4m^2 s + st + t^2}{t} \right) \,||\, \left( t < s \,\&\, \frac{4m^2(s+t) - st}{4m^2} < p_4^2 < \frac{-4m^2 t + s^2 + st}{s} \right) \right) \,\&\, m^2 > 0\,.$$

- In this region all canonical integrals and alphabet letters are real-valued, and all $l_i \in \mathcal{A}_2$ have fixed sign.
- In this region we may integrate in terms of manifestly real-valued functions

# Analytic integration

[Duhr, Gangl, Rhodes, 1110.0458]

- Generate an ansatz of basis functions, in the manner of Duhr-Gangl-Rhodes:

$$\mathrm{Li}_2\left(\pm l_i l_j\right), \mathrm{Li}_2\left(\pm \frac{l_i}{l_j}\right), \mathrm{Li}_2\left(\pm \frac{1}{l_i l_j}\right) \quad \text{for } l_i, l_j \in \mathcal{A}_2 \cup \{l_{33}, l_{38}, l_{41}\},$$

$$\log(\pm l_i)\log(\pm l_j)$$

- Require $1 - x \in \mathrm{Span}_{\mathbb{Q}}(\mathcal{A})$ for each $\mathrm{Li}_2(x)$
- Furthermore, we require $-\infty < x \leq 1$, so not to cross branch cuts of $\mathrm{Li}_2(x)$
- Then, we match the ansatz at the symbol level: $\mathcal{S}\left(B_i^{(k)}\right) = \sum_j \mathcal{S}\left(B_j^{(k-1)}\right) \otimes d\tilde{\mathbf{A}}_{ij}\,.$

# Analytic integration

- The surviving logarithmic terms are:

$$
\begin{array}{llll}
\log^2(l_1)\,, & \log^2(-l_4)\,, & \log^2(l_{25})\,, & \log^2(l_{26})\,,\\
\log^2(-l_{27})\,, & \log^2(-l_{28})\,, & \log(l_1)\log(-l_4)\,, & \log(-l_3)\log(l_{25})\,,\\
\log(-l_4)\log(l_{25})\,, & \log(-l_4)\log(l_{26})\,, & \log(-l_2)\log(-l_{27})\,, & \log(-l_5)\log(-l_{27})\,,\\
\log(-l_7)\log(-l_{27})\,, & \log(-l_8)\log(-l_{27})\,, & \log(l_{25})\log(-l_{27})\,, & \log(-l_4)\log(-l_{28})\,,\\
\log(l_9)\log(-l_{28})\,, & \log(-l_{27})\log(-l_{28})\,, & \log(l_{25})\log(l_{43})\,, & \log(l_{26})\log(l_{44})\,,\\
\log(-l_{28})\log(-l_{48})\,, & \log(-l_{28})\log(l_{55})\,, & \log(l_{26})\log(l_{56})\,, & \log(-l_{27})\log(l_{60})\,,\\
\log(-l_{27})\log(l_{61})\,. & & &
\end{array}
$$

# Analytic integration

- The surviving dilogarithms in the ansatz are:

$$\begin{array}{lllll}
\mathrm{Li}_2\left(-\frac{1}{l_{25}}\right), & \mathrm{Li}_2\left(\frac{1}{l_{25}}\right), & \mathrm{Li}_2\left(-\frac{1}{l_{26}}\right), & \mathrm{Li}_2\left(\frac{1}{l_{26}}\right), & \mathrm{Li}_2\left(-\frac{1}{l_{27}}\right), \\
\mathrm{Li}_2\left(\frac{1}{l_{27}}\right), & \mathrm{Li}_2\left(-\frac{1}{l_{25}l_{27}}\right), & \mathrm{Li}_2\left(-\frac{l_{25}}{l_{27}}\right), & \mathrm{Li}_2\left(-\frac{1}{l_{26}l_{27}}\right), & \mathrm{Li}_2\left(-\frac{l_{26}}{l_{27}}\right), \\
\mathrm{Li}_2\left(-\frac{1}{l_{28}}\right), & \mathrm{Li}_2\left(\frac{1}{l_{28}}\right), & \mathrm{Li}_2\left(\frac{1}{l_{27}l_{28}}\right), & \mathrm{Li}_2\left(\frac{l_{28}}{l_{27}}\right), & \mathrm{Li}_2\left(\frac{1}{l_{27}l_{29}}\right), \\
\mathrm{Li}_2\left(\frac{l_{29}}{l_{27}}\right), & \mathrm{Li}_2\left(-\frac{1}{l_{33}}\right), & \mathrm{Li}_2\left(-\frac{1}{l_{25}l_{33}}\right), & \mathrm{Li}_2\left(-\frac{l_{25}}{l_{33}}\right), & \mathrm{Li}_2\left(-\frac{1}{l_{26}l_{33}}\right), \\
\mathrm{Li}_2\left(-\frac{l_{26}}{l_{33}}\right), & \mathrm{Li}_2\left(\frac{1}{l_{27}l_{33}}\right), & \mathrm{Li}_2\left(\frac{l_{27}}{l_{33}}\right), & \mathrm{Li}_2\left(-\frac{1}{l_{38}}\right), & \mathrm{Li}_2\left(-\frac{1}{l_{25}l_{38}}\right), \\
\mathrm{Li}_2\left(-\frac{l_{25}}{l_{38}}\right), & \mathrm{Li}_2\left(\frac{1}{l_{27}l_{38}}\right), & \mathrm{Li}_2\left(\frac{l_{27}}{l_{38}}\right), & \mathrm{Li}_2\left(\frac{1}{l_{28}l_{38}}\right), & \mathrm{Li}_2\left(\frac{l_{28}}{l_{38}}\right), \\
\mathrm{Li}_2\left(-\frac{1}{l_{41}}\right), & \mathrm{Li}_2\left(-\frac{1}{l_{26}l_{41}}\right), & \mathrm{Li}_2\left(-\frac{l_{26}}{l_{41}}\right), & \mathrm{Li}_2\left(\frac{1}{l_{27}l_{41}}\right), & \mathrm{Li}_2\left(\frac{l_{27}}{l_{41}}\right), \\
\mathrm{Li}_2\left(\frac{1}{l_{28}l_{41}}\right), & \mathrm{Li}_2\left(\frac{l_{28}}{l_{41}}\right). & & &
\end{array}$$

- Note the spurious letters $\{l_{33}, l_{38}, l_{41}\}$.

# Analytic integration

- Solution is now determined at symbol level. Note that terms proportional to $i\pi$ are absent in the region $\mathcal{R}$.
- One also needs to fix terms like $\log(2)\log(l_i)$. Absence of such terms would show up in the differential equations. We find that it is not necessary to add these terms for any of the integrals.
- Lastly, we may fix transcendental constants from boundary conditions.

# Analytic integration

- Example, $B_{65}$ at weight 2, in region $\mathcal{R}$ is given by:

$$
\begin{aligned}
B_{65}^{(2)} = & -2\zeta_2 - 4\mathrm{Li}_2\left(-l_{27}^{-1}\right) - 4\mathrm{Li}_2\left(l_{27}^{-1}\right) - 2\mathrm{Li}_2\left(-l_{25}l_{27}^{-1}\right) + 2\mathrm{Li}_2\left(-l_{26}l_{27}^{-1}\right) + 2\mathrm{Li}_2\left(l_{28}l_{27}^{-1}\right) \\
& - 2\mathrm{Li}_2\left(-l_{25}^{-1}l_{27}^{-1}\right) + 2\mathrm{Li}_2\left(-l_{26}^{-1}l_{27}^{-1}\right) + 2\mathrm{Li}_2\left(l_{27}^{-1}l_{28}^{-1}\right) - \log^2\left(l_{25}\right) + \log^2\left(l_{26}\right) - \log^2\left(-l_{27}\right) \\
& + \log^2\left(-l_{28}\right) + 2\log\left(l_{43}\right)\log\left(l_{25}\right) - 2\log\left(l_1\right)\log\left(-l_{27}\right) + 2\log\left(-l_2\right)\log\left(-l_{27}\right) \\
& - 2\log\left(-l_5\right)\log\left(-l_{27}\right) + 2\log\left(-l_6\right)\log\left(-l_{27}\right) + 2\log\left(-l_7\right)\log\left(-l_{27}\right) \\
& - 2\log\left(-l_8\right)\log\left(-l_{27}\right) - 2\log\left(l_{26}\right)\log\left(l_{44}\right) - 2\log\left(-l_{28}\right)\log\left(-l_{48}\right)
\end{aligned}
$$

# Expressions for weight 3 and 4

- Weight 3 is directly written as a one-fold integral:

$$\vec{B}^{(i)}(\gamma(1)) = \int_\gamma d\tilde{\mathbf{A}}\vec{B}^{(i-1)} + \vec{B}^{(i)}(\gamma(0))\,.$$

- For weight 4, use an IBP-identity:

$$\begin{aligned}\vec{B}^{(i)}(\gamma(1)) &= \left[\tilde{\mathbf{A}}\vec{B}^{(i-1)}\right]_{\gamma(0)}^{\gamma(1)} - \int_\gamma \tilde{\mathbf{A}}d\vec{B}^{(i-1)} + \vec{B}^{(i)}(\gamma(0))\,,\\ &= \int_\gamma \left(\tilde{\mathbf{A}}(\gamma(1))d\tilde{\mathbf{A}} - \tilde{\mathbf{A}}d\tilde{\mathbf{A}}\right)\vec{B}^{(i-2)} + [\tilde{\mathbf{A}}]_{\gamma(0)}^{\gamma(1)}\vec{B}^{(i-1)}(\gamma(0)) + \vec{B}^{(i)}(\gamma(0))\,,\end{aligned}$$

# Boundary conditions

# Boundary conditions

- To solve the full set of differential equations, we need to compute boundary conditions for all the integrals.
- A convenient limit is the heavy mass limit / IR-limit

$$(s, t, p_4^2, m^2) \to (xs, xt, xp_4^2, m^2)\,, \quad x \to 0$$

- Asymptotic expansion (of Feynman parametrization): [See works by Beneke and Smirov]
  - Sum over all regions that do not yield scale-less contributions
  - Taylor expand the integrand in each region
  - Integrate and sum the results
- Finding the regions: asy.m See e.g. [Jantzen, Smirnov, Smirnov, 1206.0546]

# Boundary conditions

- Example:
$$B_{73} = t\epsilon^4 \left( \mathrm{I}_{1,1,1,1,1,1,1,-2,0} + \frac{4s\mathrm{I}_{1,1,1,1,1,1,1,-1,-1}}{2s+t-p_4^2} + \mathrm{I}_{1,1,1,1,1,1,1,0,-2} \right) + \\ - \frac{t\epsilon^4\left(-4s-t+p_4^2\right)}{4}\left(\mathrm{I}_{1,1,1,1,1,1,1,-1,0} + \mathrm{I}_{1,1,1,1,1,1,1,0,-1}\right) .$$
  - Elliptic
  - Closed-form in $\epsilon$ obtainable in heavy mass limit
- Asy:
$$S_1 : \quad \alpha_i \to \alpha_i \,, \\ S_2 : \quad \alpha_i \to \alpha_i \quad \text{for } i = \{1,2,4\}\,, \; \alpha_i \to x\alpha_i \quad \text{for } i = \{3,5,6,7\}\,,$$
- Scaling:
$$\lim_{x\to 0} \mathrm{I}_{1,1,1,1,1,1,1,\sigma_1,\sigma_2} \sim \mathrm{I}^{(1)}_{1,1,1,1,1,1,1,\sigma_1,\sigma_2} + x^{-\epsilon-1}\mathrm{I}^{(2)}_{1,1,1,1,1,1,1,\sigma_1,\sigma_2} \,, \\ \text{for } (\sigma_1,\sigma_2) \in \{(-2,0),(-1,0),(-1,-1),(0,-1),(0,-2)\}\,,$$

# Boundary conditions

- Hence:

$$\lim_{x\to 0} B_{73} \sim \epsilon^4 x^{-\epsilon}\left[-\frac{4st\mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,-1,-1}}{p_4^2-2s-t}+t\left(\mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,-2,0}+\mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,0,-2}\right)\right].$$

- (Terms $x^{a+b\epsilon}$ with $a > 0$ have been put to zero, since:

$$x^{a+b\epsilon} = x^a + b\,x^a \log(x)\epsilon + \frac{1}{2}b^2 x^a \log(x)^2\,\epsilon^2 + \cdots$$

and, $\lim_{x\to 0} x^a \log(x) \to 0$ for $a > 0$)

- It remains to compute the leading orders $I^{(2),(x=0)}_{1,1,1,1,1,1,1,\sigma_1,\sigma_2}$

# Boundary conditions

- We work out the example: $\mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,-2,0}$

- Symanzik polynomials:

$$\mathcal{U}^{(2),(x=0)}_{1,1,1,1,1,1,1,0,0} = (\alpha_1+\alpha_2+\alpha_4)(\alpha_3+\alpha_5+\alpha_6+\alpha_7)\;,$$

$$\mathcal{F}^{(2),(x=0)}_{1,1,1,1,1,1,1,0,0} = (\alpha_1+\alpha_2+\alpha_4)(\alpha_3+\alpha_5+\alpha_6+\alpha_7)^2 m^2 - \alpha_2\alpha_4(\alpha_3+\alpha_5+\alpha_6+\alpha_7)\, t\,.$$

- Cheng-Wu theorem: $\alpha_3 \to 1-\alpha_5-\alpha_6-\alpha_7$, $\int_0^1\int_0^{1-\alpha_7}\int_0^{1-\alpha_6-\alpha_7} d\alpha_5\, d\alpha_6\, d\alpha_7 = \frac{1}{6}\,.$

$$\begin{aligned}\mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,-2,0} = \frac{1}{6}\Gamma(2\epsilon+1)e^{2\gamma\epsilon}\left(\prod_{i\in\{1,2,4\}}\int_0^\infty d\alpha_i\right)\Big(&8(\epsilon+1)(2\epsilon+1)(m^2)^2\mathcal{F}^{-2\epsilon-3}\mathcal{U}^{3\epsilon+1}\\
&-2(2\epsilon+1)(3\epsilon-1)(\alpha_1+\alpha_2+\alpha_4)\left(2(\alpha_1+\alpha_2+\alpha_4)m^2-\alpha_2\alpha_4 t\right)\mathcal{F}^{-2\epsilon-2}\mathcal{U}^{3\epsilon-2}\\
&-8(\epsilon+1)(2\epsilon+1)m^2\alpha_2\alpha_4 t\mathcal{F}^{-2\epsilon-3}\mathcal{U}^{3\epsilon}-2(2\epsilon+1)m^2(\alpha_1+\alpha_2+\alpha_4)\mathcal{F}^{-2\epsilon-2}\mathcal{U}^{3\epsilon-1}\\
&+2(\epsilon+1)(2\epsilon+1)t^2\alpha_2^2\alpha_4^2\mathcal{F}^{-2\epsilon-3}\mathcal{U}^{3\epsilon-1}+(3\epsilon-2)(3\epsilon-1)(\alpha_1+\alpha_2+\alpha_4)^2\mathcal{F}^{-2\epsilon-1}\mathcal{U}^{3\epsilon-3}\Big)\,,\end{aligned}$$

# Boundary conditions

- Integrating out any of the remaining 3 parameters naively leads to hypergeometric ${}_2F_1$'s
- Homogenize / projectivize the integrand by letting $\alpha_i \to \alpha_i/\alpha_8$ for $i$ = 1,2,4, by including an overall $1/\alpha_8^4$ and a delta function $\delta\left(1 - \sum_{i\in\{1,2,4,8\}} \alpha_i\right)$
- Now pick the Cheng-Wu transform $\alpha_1 \to 1 - \alpha_2 - \alpha_4$
- $$\mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,-2,0} = \frac{1}{6}\Gamma(2\epsilon+1)e^{2\gamma\epsilon}\int_0^1 d\alpha_4 \int_0^{1-\alpha_4} d\alpha_2 \int_0^\infty d\alpha_8 \Big(\alpha_8^{\epsilon-1}\left(\alpha_8 m^2 - \alpha_2\alpha_4 t\right)^{-2\epsilon-3} \times \left(\alpha_8^2 m^4(\epsilon+3)(\epsilon+4) + 2\alpha_2\alpha_4\alpha_8 m^2 t(\epsilon-2)(\epsilon+4) + \alpha_2^2\alpha_4^2 t^2(\epsilon-3)(\epsilon-2)\right)\Big).$$

# Boundary conditions

- The final result is given by:

$$\mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,-2,0} = \frac{2e^{2\gamma\epsilon}\left(m^2\right)^{-\epsilon}(-t)^{-\epsilon-1}\Gamma(-\epsilon)^2\Gamma(\epsilon)\Gamma(\epsilon+2)}{(2\epsilon+2)\Gamma(1-2\epsilon)} \,.$$

where we assumed $t < 0$ during integration.

- In fact, explicit computation shows:

$$\mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,-2,0} = \mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,-1,-1} = \mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,0,-2} \,.$$

- Hence: $\lim_{x\to 0} B_{73} \sim x^{-\epsilon}\left(-4\pi e^{2\gamma\epsilon}\epsilon^3 \frac{(p_4^2-4s-t)}{(p_4^2-2s-t)}(m^2)^{-\epsilon}(-t)^{-\epsilon}\Gamma(2\epsilon)\cot(\pi\epsilon)\right)$ .

# Boundary conditions

- All boundary conditions:

$$\lim_{x\to 0} B_1 = e^{2\gamma\epsilon}\Gamma(\epsilon+1)^2(m^2)^{-2\epsilon}\,,$$

$$\lim_{x\to 0} B_2 \sim x^{-\epsilon}\left(\pi e^{2\gamma\epsilon}\epsilon(m^2)^{-\epsilon}(-t)^{-\epsilon}\Gamma(2\epsilon+1)\cot(\pi\epsilon)\right)\,,$$

$$\lim_{x\to 0} B_i = 0 \quad \text{for } i=3,\dots,72\,.$$

$$\lim_{x\to 0} B_{73} \sim x^{-\epsilon}\left(-4\pi e^{2\gamma\epsilon}\epsilon^3\frac{(p_4^2-4s-t)}{(p_4^2-2s-t)}(m^2)^{-\epsilon}(-t)^{-\epsilon}\Gamma(2\epsilon)\cot(\pi\epsilon)\right)\,.$$

- Requires computation of numerous integrals:

# Boundary conditions

```
IntSteps = Association[{
    G[6, {0, 0, 2, 0, 1, 0, 2, 0, 0}]_1 → {{"CW", α_5 → 1 - α_3}, {"Int", {α_7, 0, ∞}}, {"Int", {α_3, 0, 1}}, {"Save"}},
    G[6, {0, 0, 2, 1, 0, 0, 2, 0, 0}]_1 → {{"CW", α_7 → 1 - α_3}, {"Int", {α_4, 0, ∞}}, {"Int", {α_3, 0, 1}}, {"Save"}},
    G[6, {0, 0, 2, 1, 0, 2, 0, 0, 0}]_1 → {{"CW", α_6 → 1 - α_3}, {"Int", {α_4, 0, ∞}}, {"Int", {α_3, 0, 1}}, {"Save"}},
    G[6, {0, 0, 2, 2, 0, 1, 0, 0, 0}]_1 → {{"CW", α_6 → 1 - α_3}, {"Int", {α_4, 0, ∞}}, {"Int", {α_3, 0, 1}}, {"Save"}},
    G[6, {0, 1, 0, 0, 2, 0, 2, 0, 0}]_1 → {{"CW", α_5 → 1 - α_7}, {"Int", {α_2, 0, ∞}}, {"Int", {α_7, 0, 1}}, {"Save"}},
    G[6, {0, 1, 0, 0, 2, 2, 0, 0, 0}]_1 → {{"CW", α_6 → 1 - α_5}, {"Int", {α_2, 0, ∞}}, {"Int", {α_5, 0, 1}}, {"Save"}},
    G[6, {0, 2, 0, 0, 1, 0, 2, 0, 0}]_1 → {{"CW", α_5 → 1 - α_7}, {"Int", {α_2, 0, ∞}}, {"Int", {α_7, 0, 1}}, {"Save"}},
    G[6, {1, 0, 0, 0, 2, 0, 2, 0, 0}]_1 → {{"CW", α_5 → 1 - α_7}, {"Int", {α_1, 0, ∞}}, {"Int", {α_7, 0, 1}}, {"Save"}},
    G[6, {1, 0, 2, 0, 0, 2, 0, 0, 0}]_1 → {{"CW", α_6 → 1 - α_3}, {"Int", {α_1, 0, ∞}}, {"Int", {α_3, 0, 1}}, {"Save"}},
    G[6, {0, 0, 1, 1, 1, 0, 2, 0, 0}]_1 → {{"CW", α_3 → 1 - α_5 - α_7}, {"Int", {α_4, 0, ∞}}, {"Int", {α_5, 0, 1 - α_7}}, {"Int", {α_7, 0, 1}}, {"Save"}},
    G[6, {0, 0, 1, 1, 1, 0, 3, 0, 0}]_1 → {{"CW", α_3 → 1 - α_5 - α_7}, {"Int", {α_4, 0, ∞}}, {"Int", {α_5, 0, 1 - α_7}}, {"Int", {α_7, 0, 1}}, {"Save"}},
    G[6, {0, 0, 2, 1, 1, 0, 2, 0, 0}]_1 → {{"CW", α_3 → 1 - α_5 - α_7}, {"Int", {α_4, 0, ∞}}, {"Int", {α_5, 0, 1 - α_7}}, {"Int", {α_7, 0, 1}}, {"Save"}},
    G[6, {0, 1, 0, 1, 1, 0, 2, 0, 0}]_1 → {{"CW", α_7 → 1 - α_5}, {"Int", {α_2, 0, ∞}}, {"Int", {α_4, 0, ∞}}, {"Int", {α_5, 0, 1}}, {"Save"}},
    G[6, {0, 1, 0, 1, 1, 0, 2, 0, 0}]_2 → {{"CW", α_5 → 1 - α_7}, {"Int", {α_7, 0, 1}}, {"Resc", α_8}, {"CW", α_2 → 1 - α_4}, {"Int", {α_8, 0, ∞}}, {"Int", {α_4, 0, 1}}, {"Save"}},
    G[6, {0, 1, 1, 0, 1, 2, 0, 0, 0}]_1 → {{"CW", α_3 → 1 - α_5}, {"Int", {α_2, 0, ∞}}, {"Int", {α_5, 0, 1}}, {"Int", {α_6, 0, ∞}}, {"Save"}},
    G[6, {0, 1, 1, 0, 1, 3, 0, 0, 0}]_1 → {{"CW", α_3 → 1 - α_5}, {"Int", {α_2, 0, ∞}}, {"Int", {α_5, 0, 1}}, {"Int", {α_6, 0, ∞}}, {"Save"}},
    G[6, {0, 1, 1, 0, 2, 2, 0, 0, 0}]_1 → {{"CW", α_3 → 1 - α_5}, {"Int", {α_2, 0, ∞}}, {"Int", {α_5, 0, 1}}, {"Int", {α_6, 0, ∞}}, {"Save"}},
    G[6, {0, 1, 1, 1, 0, 2, 0, 0, 0}]_1 → {{"CW", α_6 → 1 - α_3}, {"Int", {α_4, 0, ∞}}, {"Int", {α_2, 0, ∞}}, {"Int", {α_3, 0, 1}}, {"Save"}},
    G[6, {0, 1, 1, 1, 0, 2, 0, 0, 0}]_2 → {{"CW", α_6 → 1 - α_3}, {"Int", {α_3, 0, 1}}, {"Resc", α_8}, {"CW", α_2 → 1 - α_4}, {"Int", {α_8, 0, ∞}}, {"Int", {α_4, 0, 1}}, {"Save"}},
    G[6, {0, 1, 1, 2, 0, 2, 0, 0, 0}]_2 → {{"CW", α_6 → 1 - α_3}, {"Int", {α_3, 0, 1}}, {"Resc", α_8}, {"CW", α_2 → 1 - α_4}, {"Int", {α_8, 0, ∞}}, {"Int", {α_4, 0, 1}}, {"Save"}},
    G[6, {0, 2, 2, 1, 1, 0, 0, 0, 0}]_1 → {{"CW", α_3 → 1 - α_5}, {"Int", {α_5, 0, 1}}, {"Resc", α_8}, {"CW", α_2 → 1}, {"Int", {α_8, 0, ∞}}, {"Int", {α_4, 0, ∞}}, {"Save"}},
    G[6, {1, 0, 1, 0, 1, 0, 2, 0, 0}]_1 → {{"CW", α_3 → 1 - α_5 - α_7}, {"Int", {α_1, 0, ∞}}, {"Int", {α_5, 0, 1 - α_7}}, {"Int", {α_7, 0, 1}}, {"Save"}},
    G[6, {1, 0, 1, 0, 1, 0, 3, 0, 0}]_1 → {{"CW", α_3 → 1 - α_5 - α_7}, {"Int", {α_1, 0, ∞}}, {"Int", {α_5, 0, 1 - α_7}}, {"Int", {α_7, 0, 1}}, {"Save"}},
    G[6, {1, 0, 1, 0, 1, 2, 0, 0, 0}]_1 → {{"CW", α_3 → 1 - α_5 - α_6}, {"Int", {α_1, 0, ∞}}, {"Int", {α_5, 0, 1 - α_6}}, {"Int", {α_6, 0, 1}}, {"Save"}},
    G[6, {1, 0, 1, 0, 1, 3, 0, 0, 0}]_1 → {{"CW", α_3 → 1 - α_5 - α_6}, {"Int", {α_1, 0, ∞}}, {"Int", {α_5, 0, 1 - α_6}}, {"Int", {α_6, 0, 1}}, {"Save"}},
    G[6, {1, 0, 1, 0, 2, 0, 2, 0, 0}]_1→ {{"CW", α_3 → 1 - α_5 - α_7}, {"Int", {α_1, 0, ∞}}, {"Int", {α_5, 0, 1 - α_7}}, {"Int", {α_7, 0, 1}}, {"Save"}},
    G[6, {1, 0, 2, 0, 1, 2, 0, 0, 0}]_1 → {{"CW", α_3 → 1 - α_5 - α_6}, {"Int", {α_1, 0, ∞}}, {"Int", {α_5, 0, 1 - α_6}}, {"Int", {α_6, 0, 1}}, {"Save"}},
    G[6, {1, 1, 0, 1, 1, 0, 1, 0, 0}]_2 → {{"CW", α_5 → 1 - α_7}, {"Int", {α_7, 0, 1}}, {"Resc", α_8}, {"CW", α_2 → 1 - α_4}, {"Int", {α_8, 0, ∞}}, {"Int", {α_4, 0, 1}}, {"Int", {α_1, 0, ∞}}, {"Save"}},
    G[6, {1, 1, 0, 1, 2, 0, 1, 0, 0}]_2 → {{"CW", α_5 → 1 - α_7}, {"Int", {α_7, 0, 1}}, {"Resc", α_8}, {"CW", α_2 → 1 - α_4}, {"Int", {α_8, 0, ∞}}, {"Int", {α_4, 0, 1}}, {"Int", {α_1, 0, ∞}}, {"Save"}},
    G[6, {1, 1, 1, 1, 0, 1, 0, 0, 0}]_2 → {{"CW", α_6 → 1 - α_3}, {"Int", {α_3, 0, 1}}, {"Resc", α_8}, {"CW", α_2 → 1 - α_4}, {"Int", {α_8, 0, ∞}}, {"Int", {α_4, 0, 1}}, {"Int", {α_1, 0, ∞}}, {"Save"}},
    G[6, {1, 1, 1, 1, 0, 2, 0, 0, 0}]_2 → {{"CW", α_6 → 1 - α_3}, {"Int", {α_3, 0, 1}}, {"Resc", α_8}, {"CW", α_2 → 1 - α_4}, {"Int", {α_8, 0, ∞}}, {"Int", {α_4, 0, 1}}, {"Int", {α_1, 0, ∞}}, {"Save"}},
    G[6, {1, 1, 1, 1, 1, 1, 1, -2, 0}]_2 → {{"CW", α_3 → 1 - α_5 - α_6 - α_7}, {"Int", {α_5, 0, 1 - α_6 - α_7}}, {"Int", {α_6, 0, 1 - α_7}}, {"Int", {α_7, 0, 1}}, {"Resc", α_8}, {"CW", α_1 → 1 - α_2 - α_4}, {"Int", {α_8, 0, ∞}}, {"Int", {α_2, 0, 1 - α_4}}, {"Int", {α_4, 0, 1}}, {"Save"}},
    G[6, {1, 1, 1, 1, 1, 1, 1, -1, -1}]_2 → {{"CW", α_3 → 1 - α_5 - α_6 - α_7}, {"Int", {α_5, 0, 1 - α_6 - α_7}}, {"Int", {α_6, 0, 1 - α_7}}, {"Int", {α_7, 0, 1}}, {"Resc", α_8}, {"CW", α_1 → 1 - α_2 - α_4}, {"Int", {α_8, 0, ∞}}, {"Int", {α_2, 0, 1 - α_4}}, {"Int", {α_4, 0, 1}}, {"Save"}},
    G[6, {1, 1, 1, 1, 1, 1, 1, 0, -2}]_2 → {{"CW", α_3 → 1 - α_5 - α_6 - α_7}, {"Int", {α_5, 0, 1 - α_6 - α_7}}, {"Int", {α_6, 0, 1 - α_7}}, {"Int", {α_7, 0, 1}}, {"Resc", α_8}, {"CW", α_1 → 1 - α_2 - α_4}, {"Int", {α_8, 0, ∞}}, {"Int", {α_2, 0, 1 - α_4}}, {"Int", {α_4, 0, 1}}, {"Save"}}
  }];
```

# Elliptic sectors

# Maximal cut

- There are two genuinely elliptic sectors. Their associated maximal cuts are:

$$I_{011111100} \to \int \frac{dz}{(p_4^2-t)\sqrt{z\,(z+p_4^2-t)\,\left(z^2+(p_4^2-t)z-4m^2t\right)}}$$

$$I_{111111100} \to \int \frac{dz}{t\,(z+s)\sqrt{z\,(z+p_4^2-t)\,\left(z^2+(p_4^2-t)z-4m^2t\right)}}$$

- Both cuts evaluate to elliptic integrals. The same elliptic curve appears twice. Possibly this indicates we may express all integrals in terms of iterated integrals associated to this elliptic curve.
- For now, we will solve the elliptic sectors using series expansions.

# Series expansions

- When a solution in terms of known functions is not available, it is often possible to find a series expansion which provides a good approximation.
- For single scale integrals, see for example:


S. Pozzorini and E. Remiddi, *Precise numerical evaluation of the two loop sunrise graph master integrals in the equal mass case*, *Comput. Phys. Commun.* **175** (2006) 381–387, [hep-ph/0505041].

U. Aglietti, R. Bonciani, L. Grassi, and E. Remiddi, *The Two loop crossed ladder vertex diagram with two massive exchanges*, *Nucl. Phys.* **B789** (2008) 45–83, [arXiv:0705.2616].

R. Mueller and D. G. Öztürk, *On the computation of finite bottom-quark mass effects in Higgs boson production*, *JHEP* **08** (2016) 055, [arXiv:1512.08570].

B. Mistlberger, *Higgs boson production at hadron colliders at $N^3LO$ in QCD*, *JHEP* **05** (2018) 028, [arXiv:1802.00833].

R. N. Lee, A. V. Smirnov, and V. A. Smirnov, *Solving differential equations for Feynman integrals by expansions near singular points*, *JHEP* **03** (2018) 008, [arXiv:1709.07525].

R. N. Lee, A. V. Smirnov, and V. A. Smirnov, *Evaluating elliptic master integrals at special kinematic values: using differential equations and their solutions via expansions near singular points*, *JHEP* **07** (2018) 102, [arXiv:1805.00227].

R. Bonciani, G. Degrassi, P. P. Giardino, and R. Gröber, *A Numerical Routine for the Crossed Vertex Diagram with a Massive-Particle Loop*, *Comput. Phys. Commun.* **241** (2019) 122–131, [arXiv:1812.02698].

# Series expansions

- For multi-scale problems, series expansions have been considered in special kinematic limits:

K. Melnikov, L. Tancredi, and C. Wever, *Two-loop $gg \to Hg$ amplitude mediated by a nearly massless quark*, *JHEP* **11** (2016) 104, [arXiv:1610.03747].

K. Melnikov, L. Tancredi, and C. Wever, *Two-loop amplitudes for $qg \to Hq$ and $q\bar{q} \to Hg$ mediated by a nearly massless quark*, *Phys. Rev.* **D95** (2017), no. 5 054012, [arXiv:1702.00426].

R. Bonciani, G. Degrassi, P. P. Giardino, and R. Grober, *Analytical Method for Next-to-Leading-Order QCD Corrections to Double-Higgs Production*, *Phys. Rev. Lett.* **121** (2018), no. 16 162003, [arXiv:1806.11564].

R. Bruser, S. Caron-Huot, and J. M. Henn, *Subleading Regge limit from a soft anomalous dimension*, *JHEP* **04** (2018) 047, [arXiv:1802.02524].

J. Davies, G. Mishima, M. Steinhauser, and D. Wellmann, *Double-Higgs boson production in the high-energy limit: planar master integrals*, *JHEP* **03** (2018) 048, [arXiv:1801.09696].

J. Davies, G. Mishima, M. Steinhauser, and D. Wellmann, *Double Higgs boson production at NLO in the high-energy limit: complete analytic results*, *JHEP* **01** (2019) 176, [arXiv:1811.05489].

# Series expansions

- For our family of integrals, we follow a general approach discussed in more detail in F. Moriello, *Generalised power series expansions for the elliptic planar families of Higgs + jet production at two loops*, arXiv:1907.13234.

- Main steps:
  - Write down a sequence of paths / lines to a kinematic point.
  - Series expand the differential equations along each path
  - Solve the differential equations in terms of (generalized) series, along each path, and use the result to fix the boundary conditions for the next path.

# Series expansions

- Polylogarithmic case: $d\vec{B} = \epsilon d\tilde{\mathbf{A}}\vec{B}\,,$
- Order-by-order along a path $\gamma(\lambda) : [0,1] \to \mathbb{C}^4$

$$\frac{\partial \vec{B}^{(i)}(\lambda)}{\partial \lambda} = \mathbf{A}_\lambda \vec{B}^{(i-1)}(\lambda).$$

$$\vec{B}^{(i)}(\gamma(1)) = \int_0^1 \mathbf{A}_\lambda \vec{B}^{(i-1)} d\lambda + \vec{B}^{(i)}(\gamma(0))\,.$$

- Series expand $\mathbf{A}_\lambda$, then each integration is of the form

$\int \lambda^w \log(\lambda)^n\,,$ for $w \in \mathbb{Q}$ and $n \in \mathbb{Z}_{\geq 0}$

whose primitives lie in the same class, for example:

$$\int \lambda^{-3/5} \log^2(\lambda) d\lambda = \tfrac{5}{4}\lambda^{2/5}\left(2\log^2(\lambda) - 10\log(\lambda) + 25\right)$$

# Series solutions for higher order deqn's

- Consider a differential equation of the form

$$\frac{\partial^k g(\lambda)}{\partial \lambda^k} + a_1(\lambda)\frac{\partial^{k-1} g(\lambda)}{\partial \lambda^{k-1}} + \cdots + a_k(\lambda) g(\lambda) = \beta(\lambda)$$

- Then a general solution is given by

$$g(\lambda) = \sum_{j=1}^{k} \left( c_j g_j^h(\lambda) + g_j^h(\lambda) \int \frac{W_j(\lambda)}{W(\lambda)} ) d\lambda \right),$$

Where $c_j$ denote constants to be fixed from boundary conditions,

$$W(\lambda) = \begin{vmatrix} g_1^h(\lambda) & g_2^h(\lambda) & \cdots & g_n^h(\lambda) \\ g_1^{h\prime}(\lambda) & g_2^{h\prime}(\lambda) & \cdots & g_n^{h\prime}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ g_1^{h,(n-1)}(\lambda) & g_2^{h,(n-1)}(\lambda) & \cdots & g_n^{h,(n-1)}(\lambda) \end{vmatrix} \quad W_j(\lambda) = \begin{vmatrix} g_1^h(\lambda) & \cdots & 0 & \cdots & g_n^h(\lambda) \\ g_1^{h\prime}(\lambda) & \cdots & 0 & \cdots & g_n^{h\prime}(\lambda) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_1^{h,(n-1)}(\lambda) & \cdots & \underbrace{\beta(\lambda)}_{j\text{-th pos}} & \cdots & g_n^{h,(n-1)}(\lambda) \end{vmatrix}$$

# Homogeneous solutions: Frobenius method

- Frobenius method: general method for finding series solutions in the vicinity of a regular (singular) point.

Textbook discussion: [Coddington, 1955]

- Relies on an ansatz $g_r^h(\lambda) = \lambda^r \sum_{i=0}^{\infty} g_{r,i}^h \lambda^i$.
- First series coefficient gives "indicial equation" $P(r) = 0$, which fixes possible values for $r$.
- If the roots of $P(r)$ don't differ by integers, each value of $r$ gives a different homogeneous solution.

# Homogeneous solutions: Frobenius method

- In some cases there are roots of the indicial equation which differ by integers. In these cases the ansatz only works for the largest root.
- We illustrate for the degree 2 case, how to obtain the second root:

$$\frac{\partial^2 f(\lambda)}{\partial \lambda^2} + a_1(\lambda)\frac{\partial f(\lambda)}{\partial \lambda} + a_2(\lambda) f(\lambda) = 0$$

- Suppose $f_h(\lambda)$ is a homogeneous solution.
- Consider $\tilde{f}(\lambda) = \mu(\lambda) f_h(\lambda)$ and plug it into the differential equation. The resulting equation takes the form:

$$\frac{\partial \mu(\lambda)}{\partial \lambda}\left(a_1 + 2\frac{1}{f_h(\lambda)}\frac{\partial f_h(\lambda)}{\partial \lambda}\right) + \frac{\partial^2 \mu(\lambda)}{\partial \lambda^2} = 0$$

# Homogeneous solutions: Frobenius method

- Hence, we obtain a first order homogeneous differential equation for $\mu(\lambda)'$.
- We may directly write down the general solution:

$$\mu(\lambda) = \int \frac{1}{f_h(\lambda)^2} e^{-\int a_1(\lambda) d\lambda} d\lambda$$

and the second solution is given by $f_2^h(\lambda) = \mu(\lambda) f_1^h(\lambda)$ .

- We have now solved the 2nd order differential equation, in terms of a series with 2 remaining unfixed parameters that multiply the homogeneous solutions. These may lastly be fixed by boundary conditions.

# Precision and integration order

- The precision of the method is affected by the convergence radius of each expansion, and the order that is chosen for the expansion.
- A rule of thumb is to evaluate each expansion halfway along the distance to its nearest singularity.
- Additional ways to estimate precision of each expansion are:
  - Solving the differential equations up to different orders and taking the difference of the results.
  - Comparing the difference between the expanded matrix elements of the differential equations, with the expanded differential equations.
- Best estimate of precision: integrate along 2 different paths to the same point.

# Series solutions of family F

- We consider the following choice of basis of the elliptic sectors:

$$B_{66} = s\epsilon^4 r_2 \mathrm{I}_{0,1,1,1,1,1,1,0,0}\,,$$

$$B_{67} = \epsilon^4 r_2 \mathrm{I}_{-2,1,1,1,1,1,1,0,0}\,,$$

$$B_{68} = t\epsilon^4 \left(p_4^2 - t\right)\left(\mathrm{I}_{1,1,1,1,1,1,1,-1,0} - \mathrm{I}_{1,1,1,1,1,1,1,0,-1}\right)\,,$$

$$B_{69} = t\epsilon^4 \left(\mathrm{I}_{1,1,1,1,1,1,1,-2,0} - \mathrm{I}_{1,1,1,1,1,1,1,0,-2} + s\left(\mathrm{I}_{1,1,1,1,1,1,1,-1,0} - \mathrm{I}_{1,1,1,1,1,1,1,0,-1}\right)\right)\,,$$

$$B_{70} = t\epsilon^4 r_{16}\left(\mathrm{I}_{1,1,1,1,1,1,1,-1,0} + \mathrm{I}_{1,1,1,1,1,1,1,0,-1}\right)\,,$$

$$B_{71} = \frac{t\epsilon^4 \left(p_4^2 - t\right)^2}{\left(2s + t - p_4^2\right) r_{16}} \mathrm{I}_{1,1,1,1,1,1,1,-1,-1}\,,$$

$$B_{72} = t\epsilon^4 r_2 r_5 r_{12} \mathrm{I}_{1,1,1,1,1,1,1,0,0}\,,$$

$$B_{73} = t\epsilon^4 \Bigg(\mathrm{I}_{1,1,1,1,1,1,1,-2,0} + \frac{4s}{-p_4^2 + 2s + t}\mathrm{I}_{1,1,1,1,1,1,1,-1,-1} + \mathrm{I}_{1,1,1,1,1,1,1,0,-2} + \\ + \frac{1}{4}\left(4s + t - p_4^2\right)\left(\mathrm{I}_{1,1,1,1,1,1,1,-1,0} + \mathrm{I}_{1,1,1,1,1,1,1,0,-1}\right)\Bigg)$$

# Series solutions of family F

- The differential equations are:

$$\frac{\partial}{\partial x_i}\vec{B}_{66-73}(\vec{x},\epsilon) = \sum_{j=0}^{\infty}\epsilon^j \mathbf{A}_{x_i}^{(j)}(\vec{x})\vec{B}_{66-73}(\vec{x},\epsilon) + \vec{G}_{66-73}(\vec{x},\epsilon)$$

- The homogeneous matrix has the following schematic form:

$$\mathbf{A}_{\lambda}^{(0)} = \left(\begin{array}{cc|cccccc} 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & * & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & 0 & 0, \end{array}\right)$$

- Hence, integrals 66,67 and 70,71 are coupled.

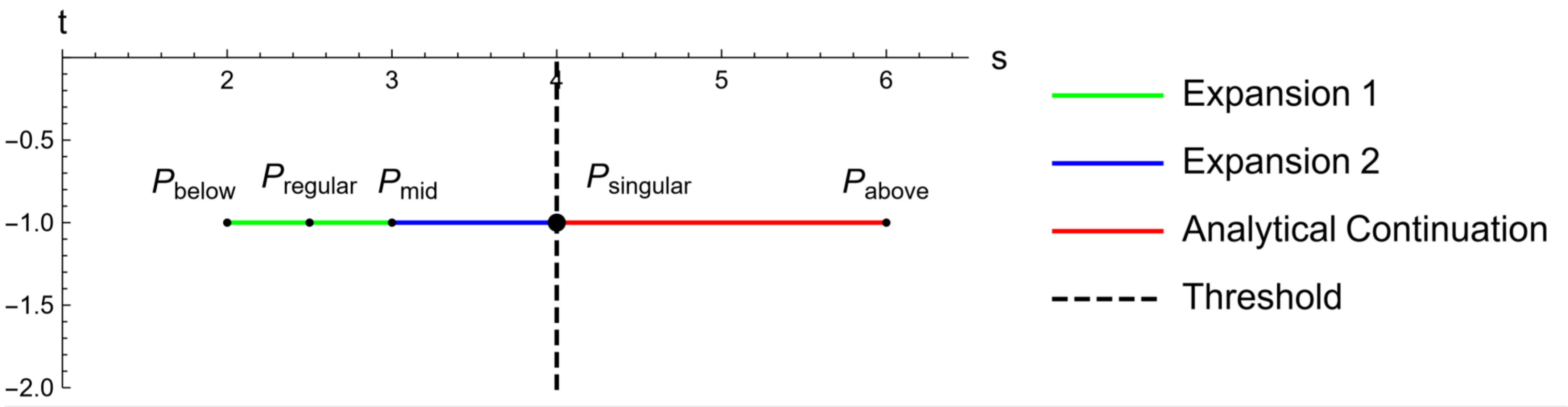
# Series solutions of family F

- Example: We consider a path $\mathrm{P}_{\text{below}} = (s = 2, t = -1, p_4^2 = 13/25, m^2 = 1)$

$$\downarrow \gamma$$

$$\mathrm{P}_{\text{above}} = (s = 6, t = -1, p_4^2 = 13/25, m^2 = 1)$$

which crosses a particle production threshold. Along the path we defined two expansions, one centered at $P_{\text{regular}}$ and one at $P_{\text{singular}}$, which are matched at $P_{\text{mid}}$:



# Series solutions of family F

- In addition we use 10 expansions, to reach $P_{\text{regular}}$, starting from the heavy mass limit

$$(s, t, p_4^2, m^2) \to (\lambda s, \lambda t, \lambda p_4^2, m^2)\,, \quad \lambda \to 0$$

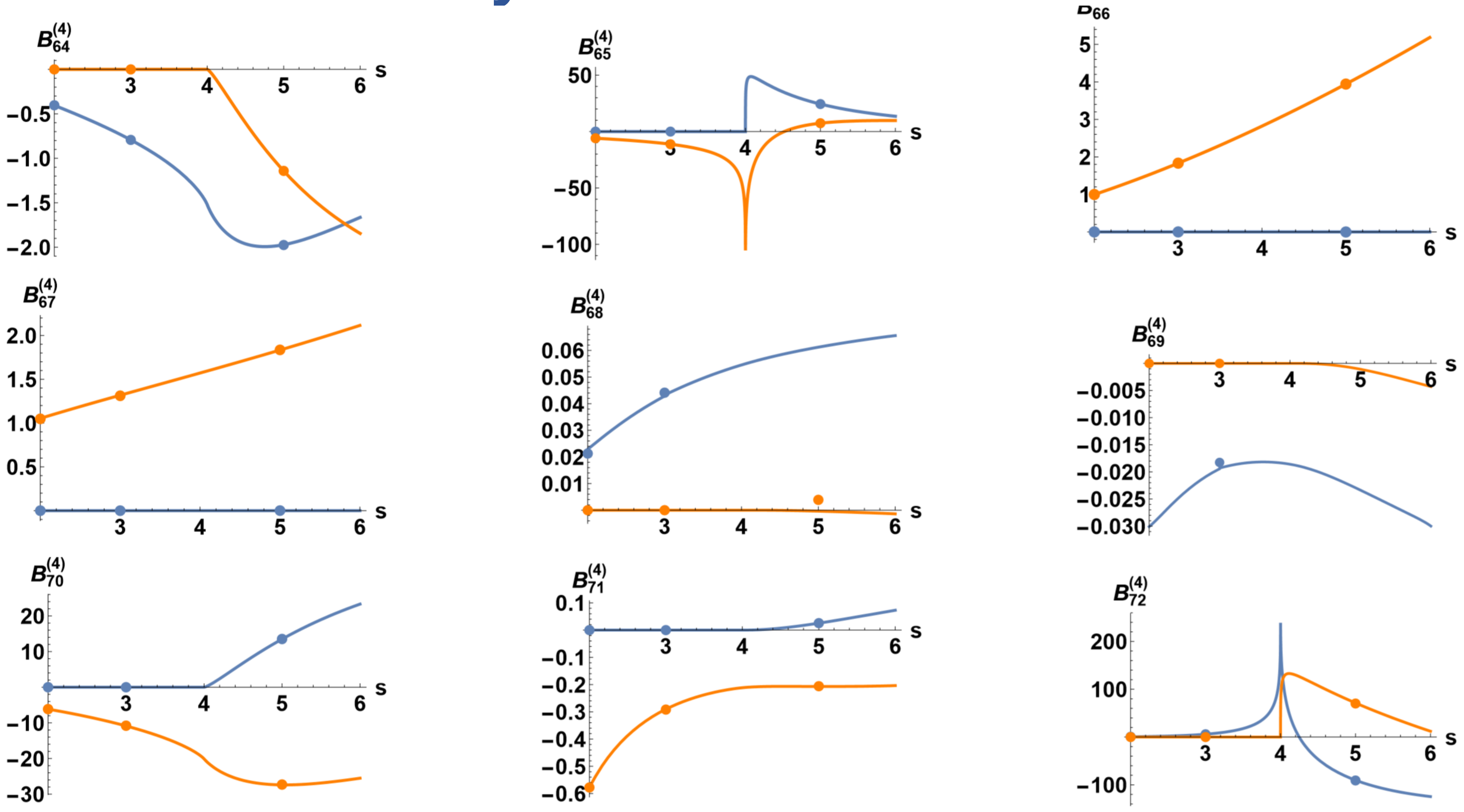
- The analytic continuation from $P_{\text{mid}}$ to points past the threshold up to $P_{\text{above}}$ is done using the $i\delta$ prescription, which tells use to assign a small positive imaginary part to $s$. This in turn tells us which branch to pick for the logarithms $\log(\lambda)$. For example, if our line points in the direction where $s$ decreases, we consider:

$$\log(\lambda - i\delta) = \theta(\lambda)\log(\lambda) + \theta(-\lambda)(\log(-\lambda) - i\pi)$$

- Thus the analytic continuation is rendered trivial!

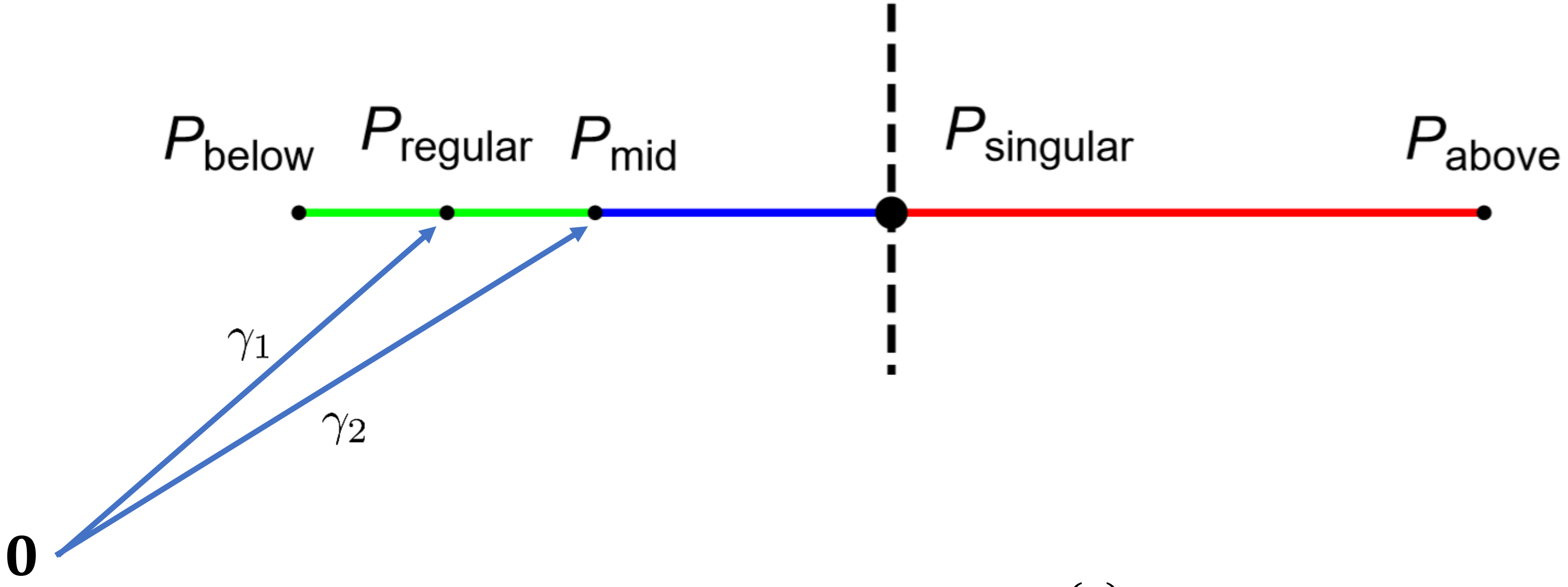
# Plots for family F

The real part of the integrals is in blue, the imaginary part is orange.



# Consistency check

- Consider another sequence of expansions from the heavy mass limit directly to $P_{\text{mid}}$




(Schematic)

- Largest difference in the integrals at $P_{\text{mid}}$ is for $B_{72}^{(4)}$ and of order $10^{-16}$
- Finding the expansion above threshold takes around $\mathcal{O}(100s)$, i.e. $\mathcal{O}(1s)$ per integral, on a single core. On the other hand FIESTA takes $\mathcal{O}(10^4 s)$ on 48 cores.

# Last remarks on series expansions

- The precision can be increased further by performing more expansions and/or increasing the order of the expansions.

| | Truncation | Relative error | Total time (73 MIs) | Time/integral |
|---|---|---|---|---|
| Expansion | $\mathcal{O}(t^{85})$ | $\leq 10^{-24}$ | 79 sec | 1.1 sec |
| Expansion | $\mathcal{O}(t^{125})$ | $\leq 10^{-32}$ | 162 sec | 2.2 sec |
| FIESTA 4.1 | | $10^{-3}$ | $\mathcal{O}(10000)$ sec | $\mathcal{O}(100)$ sec |

Table taken from F. Moriello's paper arXiv:1907.13234, on expansions of the planar Higgs + jet families, along a similar path crossing a threshold.

- Some preliminary tests with Padé approximants show even better precision may be reached from the same expansions.
- The expansion strategy is also expected to work for Feynman integrals beyond elliptic type, such as banana graphs.
- The method is suitable for automation in a software / Mathematica package.

# Conclusion and outlook

- We computed a family of non-planar master integrals relevant for Higgs + jet production at NLO including full heavy quark mass dependence.
- The given family has 73 master integrals, with elliptic top sectors.
- Polylogarithmic sectors: we derived a canonical basis, and an alphabet of 69 letters, with 10 different square roots. We performed an analytic integration at weight 2.
- We obtained series expansions valid at high precision for all integrals (including the elliptic sectors), both below and above a particle production threshold.
- Outlook: compute remaining non-planar family, fully automate the expansions.

# Thank you for listening!