Solutions to the General Relativistic Two-Body Problem in the Post-Newtonian Approximation Scheme: A Review

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Abstract

The steady increase in sensitivity in observational astrophysics requires theoreticians to improve the precision of their predictions. Especially the latest improvements in detection sensitivity of gravitational waves from binary systems requires more and more accuracy in the description of relativistic effects in such systems. Binary systems are described by a classical, celestial mechanics problem, namely the two-body problem. In this work I review some of the efforts made to solve this problem, starting with the discussion of the Newtonian two-body problem. In the following, the Post-Newtonian approximation scheme is introduced, which constitutes the framework for solving the relativistic problem. The solutions up to the third order of the Post-Newtonian approximation scheme, as well as their derivations, are discussed. The solutions are tweaked such that they resemble the classical Keplerian solution most closely. Future experiments will probably require the solution of the problem at even higher orders.

I Introduction

The sensitivity of experiments aiming at measuring general relativistic effects in extraterrestrial systems is increasing at a steady pace. In order to keep up with this development theoreticians are required to improve the accuracy of their predictions. A prime example of such a development are the latest successes in the field of detection of gravitational waves.
Many of those experiments are studying the properties of systems consisting of two compact objects that interact by gravity, so called binaries. At the moment there are two types of such experiments. The first one investigates the emission of gravitational waves from inspiraling black holes and the second one the change of the orbital period in neutronstar-pulsar systems. Since such measurements provide powerful tests of general relativity there is a large interest in calculating the properties of such binary systems with high precision. Since the Einstein field equations are non-linear there is no general, analytic solution of them available and one has to rely on approximative methods or simple solutions. One of the approximative methods used to deal with those equations is the Post-Newtonian approximation scheme [4].

In this review I plan to summarise some of the calculations performed on compact binaries in the Post-Newtonian approximation scheme up to third order.
In section II the Newtonian two-body problem is discussed. The derivation is shown in detail since it constitutes the basic calculation scheme followed in higher order calculations as well. The next section introduces the principles of the Post-Newtonian approximation scheme. Section IV treats the relativistic two-body problem considering only first order contributions from relativistic effects. To illustrate the principle of the derivation the steps needed are discussed in detail. In section V however, where the problem is treated at second and third order, the derivations are not outlined in detail since the algebra at those orders becomes very tedious and lengthy. Only the most important steps are discussed and the results are presented. However, some computational details are outlined in Appendix B. The last section summarises the discussed topics.

II The Newtonian Two-Body Problem

Being one of the most heavily studied problems in celestial mechanics the Newtonian two-body problem dealing with two compact objects interacting with each other only by gravity, was solved in several different ways. In the event of the following discussion of the relativistic analogue of the problem I present the one derivation that is used in a similar way to solve the relativistic problem below. The discussion of the Newtonian Two-Body Problem in this chapter is based entirely on Appendix A in the work on general relativistic celestial mechanics of binary systems by T. Damour and N. Deruelle [6].

In a first approximation the spatial extend of the objects is neglected and one only considers point-masses. Recalling that in Newtonian physics the gravitational potential of two point masses is given by $V = -\frac{\mathrm{G}mm'}{|\vec{r}-\vec{r}'|}$ the Lagrangian of the system can easily be derived:

$$L_N = \frac{1}{2}(mv^2 + m'v'^2) + \frac{\mathrm{G}mm'}{|\vec{r}-\vec{r}'|}. \tag{1}$$

In equation (1) and the following discussion G denotes the Newtonian gravitational constant, $\vec{r}, \vec{v}$ and m the positional and velocity vectors as well as the mass of the first particle. The primed variables denote the same quantities but for the second particle. Noether's theorem dictates the conservation of the total momentum $\vec{P}_N$ as well as the centre-of-mass integral $\vec{K}_N$ of the system due to symmetry of the Lagrangian under spatial translations (conservation of total momentum) and Galilei transformations (conservation of center-of-mass-integral):

$$\vec{P}_N = m\vec{v} + m'\vec{v}' \tag{2}$$

$$\vec{K}_N = m\vec{r} + m'\vec{r}' - t\vec{P}_N \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{P}_N = \frac{\mathrm{d}}{\mathrm{d}t}\vec{K}_N = 0 \tag{4}$$

Performing a coordinate transformation to the center-of-mass frame of the system such that

$$\vec{r} = \frac{\mu}{m}\vec{R}, \qquad \vec{r}' = -\frac{\mu}{m'}\vec{R} \tag{5}$$

$$\vec{v} = \frac{\mu}{m}\vec{V}, \qquad \vec{v}' = -\frac{\mu}{m'}\vec{V} \tag{6}$$

renders $\vec{P}_N = \vec{K}_N = 0$ and allows to rewrite equation (1) using relative coordinates:

$$L_N = \frac{1}{2}\mu V^2 + \frac{\mathrm{G}mm'}{R} \tag{7}$$

Where the relative coordinates $\vec{R} = \vec{r} - \vec{r}'$ and $\vec{V} = \vec{v} - \vec{v}'$ as well as the reduced mass $\mu = \frac{mm'}{m+m'}$ were introduced. Following the principle of least action

$$0 = \frac{\delta S_N}{\delta \vec{R}} = \int \frac{\delta L_N}{\delta \vec{R}} \mathrm{d}t = \int \left(\frac{\partial}{\partial \vec{R}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \vec{V}} \right) L_N \, \mathrm{d}t \tag{8}$$

one arrives at the equations of motion

$$\frac{\mathrm{d}\vec{V}}{\mathrm{d}t} = -\frac{\mathrm{G}M}{R} \vec{N}. \tag{9}$$

$M = m + m'$ denotes the total mass of the system and $\vec{N} = \vec{R}/R$ the unit vector along the relative position vector. In principle, one could proceed and try to find the parametrisation of the orbit using the equations of motion. A much easier approach is to consider the total energy E and total angular momentum $\vec{J}$ of the system. Noether's theorem predicts those to be constants of motion due to symmetry of the Lagrangian under time translations and space rotations. They are given as:

$$E = \frac{1}{2}V^2 - \frac{\mathrm{G}M}{R}, \tag{10}$$

$$\vec{J} = \vec{R} \wedge \vec{V}. \tag{11}$$

Since the conservation of the total angular momentum implies the motion of the system to take place in a plane it is most simple to work in polar coordinates. Therefore, $\vec{R} = R \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$ and one can use the relations

$$V^2 = \left(\frac{\mathrm{d}R}{\mathrm{d}t} \right)^2 + R^2 \left(\frac{\mathrm{d}\theta}{\mathrm{d}t} \right)^2, \tag{12}$$

$$|\vec{R} \wedge \vec{V}| = R^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{13}$$

together with equations (10) and (11) to arrive at

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t} \right)^2 = A + \frac{2B}{R} + \frac{C}{R^2}, \tag{14}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{H}{R^2}. \tag{15}$$

Where the coefficients are easily computed to be $A = 2E$, $B = \mathrm{G}M$, $C = -J^2$, $H = |\vec{J}|$.

The solutions of equations (14) and (15) differ depending on the value of the coefficient A. In the case $A < 0$ one finds an elliptic motion:

$$R = a(1 - e\cos(u)), \tag{16}$$

$$\theta - \theta_0 = A_e(u) \equiv 2 \arctan \left(\sqrt{\frac{1+e}{1-e}} \tan\left(\frac{u}{2}\right) \right) \tag{17}$$

The solutions are given as a function of the eccentric anomaly u (see Figure 1). The newly introduced parameters a and e are called semi major axis and eccentricity.
For a detailed derivation of equations (16) and (17) see Appendix A. One also finds the time dependence of the eccentric anomaly u to be

$$n(t - t_0) = u - e\sin(u). \tag{18}$$

Where the mean motion $n = \sqrt{\frac{B}{a^3}}$ is used to encapsulate the period of the system by $T = 2\pi/n$. Equation (18) is usually called Kepler's equation.

As the system performs a full orbit the periastron (or perihelion respectively) moves by $\Delta\theta = 2\pi$. I stress this result here since this will be different in the general relativistic case where one finds a precession of the

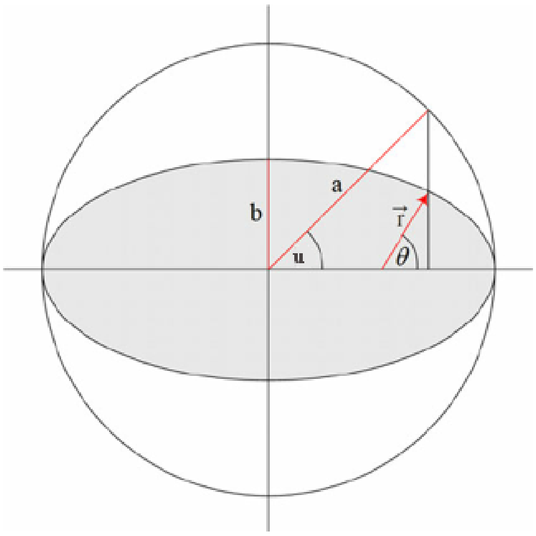

Figure 1: Illustration of the difference between eccentric anomaly u and true anomaly θ, based on [10].

ellipse.

Eliminating the eccentric anomaly u in favour of the true anomaly θ in equation (16) by using equation (17) one finds the orbital equation

$$R = \frac{a(1-e^2)}{1+e\cos(\theta-\theta_0)}. \tag{19}$$

The orbits of the two point-masses are obtained easily by using equations (5) and (6):

$$r = \frac{\mu}{m}\frac{a(1-e^2)}{(1+e\cos(\theta-\theta_0))}, \tag{20}$$

$$r' = -\frac{\mu}{m'}\frac{a(1-e^2)}{(1+e\cos(\theta-\theta_0))} \tag{21}$$

One also notices that the orbital angle of the first point-mass is equal to the relative orbital angle θ and that the orbital angle of the second point-mass is given by $\theta+\pi$.

The case $A > 0$ describing hyperbolic motion can be obtained by analytic continuation of the elliptic solution given above. Using the replacements $u = iv, \bar{a} = -a$ and $\bar{n} = -in$ one easily computes:

$$R = \bar{a}(e\cosh(v) - 1), \tag{22}$$

$$\theta - \theta_0 = 2\arctan\left(\sqrt{\frac{1+e}{1-e}}\tanh\left(\frac{v}{2}\right)\right) \tag{23}$$

and Kepler's equation becomes

$$\bar{n}(t-t_0) = e\sinh(v) - v. \tag{24}$$

The limiting case A=0 is obtained in an even simpler fashion. One just considers the elliptic case and takes the limit of $A \to 0$. Exemplary orbits illustrating the differences between the three cases are shown in Figure 2.

In the following discussion only the elliptic cases are studied. The hyperbolic and limiting cases can be obtained in an analogous fashion as in the non-relativistic case.

III The Post-Newtonian Approximation for binary systems

The introduction of the Post-Newtonian framework as presented in this section is based on the lecture notes on Applications of General Relativity by Prof. Jetzer [8] and the book by N. Straumann [12].

The Post-Newtonian Approximation (PN) is well suited for the treatment of moderately relativistic systems. It was introduced to describe systems consisting of multiple compact objects in which energy loss by gravitational radiation can be neglected. The basic principle is that the Einstein field equations, which do not posses an

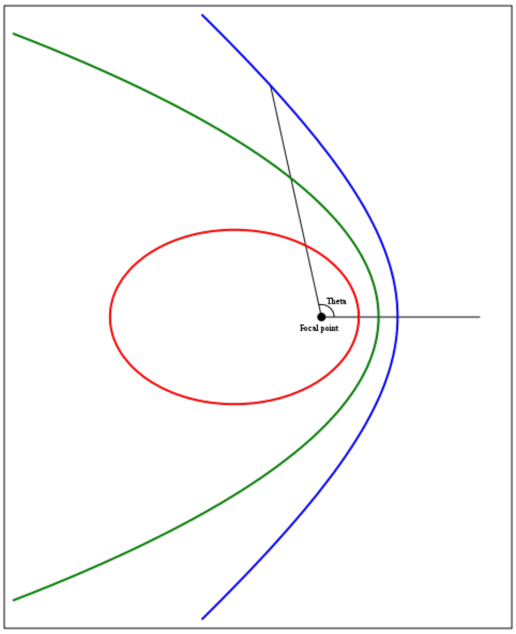

Figure 2: Exemplary orbits for the three different solutions of the Newtonian two-body problem. Blue: hyperbolic ($A > 0$), Red: elliptic $A < 0$ and Green: parabolic $A = 0$ [2].

analytical solution for such a system, are expanded in some small quantity. Such systems are characterised by slow motion of the constituents ($v/c << 1$) and weak gravity ($\mathrm{G}M/R\mathrm{c}^2 << 1$). Therefore, it is most convenient to expand the equations in powers of some small quantity $\epsilon \sim \frac{v}{c} \sim \sqrt{\frac{\mathrm{G}M}{R\mathrm{c}^2}} \sim \mathrm{c}^{-1}$.

When studying the motion of a binary system one would like to work either in the Hamiltonian or in the Lagrangian framework. I illustrate how to tackle the problem in the Lagrangian framework here. The problem can be divided in two subproblems. 1.) Finding the equations of motion and 2.) Solving them [6]. The second subproblem will be discussed in the following chapters. Therefore, let us focus on the first subproblem in this chapter. Considering a binary system the action is calculated from the Lagrangian of the system:

$$S = \sum_{i=1,2} S_i = \int \mathrm{d}t \, L_0 + \frac{1}{\mathrm{c}^2} L_1 + \frac{1}{\mathrm{c}^4} L_2 + ... \tag{25}$$

where the Lagrangian has already been split up into its different orders and the summation on the left hand side of the equation is performed over the two constituents of the system. The actions S_i are given by

$$S_i = -m_i \mathrm{c} \int \sqrt{-g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \frac{\mathrm{d}x^\nu}{\mathrm{d}t}} \, \mathrm{d}t. \tag{26}$$

Where I introduced the metric tensor $g_{\mu\nu}$. The components of the metric tensor are expanded as:

$$g_{00} = -1 + g_{00}^{(2)} + g_{00}^{(4)} + g_{00}^{(6)} + ..., \tag{27}$$

$$g_{0i} = g_{0i}^{(3)} + g_{0i}^{(5)} + ..., \tag{28}$$

$$g_{ij} = \delta_{ij} + g_{ij}^{(2)} + g_{ij}^{(4)} + ... \tag{29}$$

The number in brackets indicates the order of c^{-1} in the different terms. The blue terms are the Newtonian parts of the components whereas the red terms constitute the first Post-Newtonian terms (1PN) and the green ones are the second Post-Newtonian terms (2PN). Notice that in this scheme the Einstein field equation is expanded to a certain order. The metric components are accompanied by derivatives in the Einstein field equation which contribute further orders of c^{-1}. Therefore the different metric components do not contain the same number of c^{-1} factors although belonging to the same order of the PN expansion.

Inserting the expanded metric components as well as the expansion of the energy-momentum tensor $T_{\mu\nu}$ into the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{8\pi \mathrm{G}}{\mathrm{c}^4} T_{\mu\nu} \tag{30}$$

one obtains relations connecting the different metric components with the energy-momentum tensor components. $R_{\mu\nu}$ denotes the Ricci curvature tensor. Note also that the minus sign in the Einstein field equation depends on the convention used in the metric tensor. The energy-momentum tensor of a single point-mass particle can be written as

$$T_i^{\mu\nu} = \frac{1}{\sqrt{-\det(g)}} \gamma_i m_i \frac{\mathrm{d}x_i^\mu}{\mathrm{d}t} \frac{\mathrm{d}x_i^\nu}{\mathrm{d}t} \delta^{(3)}(\vec{x} - \vec{x_i}) \tag{31}$$

where $\vec{x_i}$ denotes the spatial position of the particle and $\gamma_i = \frac{\mathrm{d}\tau_i}{\mathrm{d}t} = \left(-g_{00} - g_{lj} v_i^j v_i^l / \mathrm{c}^2\right)^{-1/2}$. Expanding γ_i and $\det(g)$ in the above equation in powers of c^{-1} one obtains an expansion of the energy momentum tensor which can then be used to solve the equations obtained from the Einstein field equations for the constituents of the metric tensor. Those constituents are then plugged into the expression for the action which allows to solve for the Lagrangian components. Depending on how many PN orders one is considering the derivation of the Lagrangian becomes very lengthy. I state only the 1PN result

$$L_{1PN} = L_N + \frac{1}{c^2} L_2. \tag{32}$$

where L_N is given in equation (1) and L_2 is found to be

$$L_2 = \frac{1}{8} m v^4 + \frac{1}{8} m' v'^4 + \frac{\mathrm{G}mm'}{2R} \left(3v^2 + 3v'^2 - 7\vec{v}\vec{v}' - (\vec{N}\vec{v})(\vec{N}\vec{v}') - \mathrm{G}\frac{m+m'}{R}\right) \tag{33}$$

where I used the same notation already introduced in the treatment of the Newtonian problem. The 1PN Lagrangian is called the Einstein-Infeld-Hoffmann Lagrangian. Using the principle of least action one can calculate the equations of motion.

Note that since we are calculating the actions of the two particles separately we only need to consider the energy-momentum tensor of a single particle since each of the particles moves in the field generated by the other particle. One should also keep in mind that we are actually dealing with large stellar objects having a finite spatial extend. However, when considering only 1PN terms we can approximate the particles as point-particles without making any noticeable errors.

IV The Two-Body Problem at 1PN

The calculation of the 1PN corrections to the two-body problem presented in this chapter follows closely the analysis presented in [6]. Note that the same notation as in the non-relativistic case is used in this chapter as well. Therefore, I omit the redefinition of the parameters that were already introduced above.

At 1PN one can use the Einstein-Infeld-Hoffmann Lagrangian for two particles stated in equation (32). As in the non-relativistic case Noether's theorem tells us that the total momentum $\vec{P}_{1PN}$(by symmetry under space translations) and the relativistic center-of-mass-integral $\vec{K}_{1PN}$ (by symmetry under Lorentz transformations) are conserved. Note that $\vec{P}_{1PN} \neq \vec{P}_N$ and $\vec{K}_{1PN} \neq \vec{K}_N$ since 1PN corrections are taken into account now. Following the same procedure as in the non-relativistic analysis one transforms to the center-of-mass system in which $\vec{P}_{1PN} = \vec{K}_{1PN} = 0$. The position vectors in that frame become

$$\vec{r} = \frac{\mu}{m}\vec{R} + \frac{\mu(m-m')}{2M^2\mathrm{c}^2}\left(V^2 - \frac{\mathrm{G}M}{R}\right)\vec{R} \equiv \frac{\mu}{m}\vec{R} + \frac{1}{\mathrm{c}^2}\vec{X}_{rel}, \tag{34}$$

$$\vec{r}' = -\frac{\mu}{m'}\vec{R} + \frac{\mu(m-m')}{2M^2\mathrm{c}^2}\left(V^2 - \frac{\mathrm{G}M}{R}\right)\vec{R} \equiv -\frac{\mu}{m'}\vec{R} + \frac{1}{\mathrm{c}^2}\vec{X}_{rel}. \tag{35}$$

Basically one could now proceed as before and replace the quantities $\vec{r}, \vec{r}', \vec{v}$ and $\vec{v}'$ in the Lagrangian in equation (32) by the relative quantities by using equations (34) and (35). Using the principle of least action one can then derive the equations of motion. However, it can be shown that it is sufficient to use the non-relativistic center-of-mass expressions of the coordinates shown already in equations (5) and (6) instead of the lengthy expressions in equations (34) and (35). This approach leads to the same equations of motion. The basic argument why this is possible is that L_N is not affected at all by this change of coordinates but L_2 is. The difference $\vec{X}_{rel}/\mathrm{c}^2$ between equations (5) and (6) and equations (34) and (35) is of order c^{-2} but since L_2 is accompanied by a c^{-2} factor in equation (32) itself the difference is of order c^{-4} and does not contribute to the Lagrangian at 1PN. A more sophisticated proof is given in [6].

Therefore, using equations (5) and (6) one writes the 1PN Lagrangian as

$$L^R_{PN} = \frac{1}{2}V^2 + \frac{\mathrm{G}M}{R} + \frac{1}{8}(1-3\nu)\frac{V^4}{\mathrm{c}^2} + \frac{\mathrm{G}M}{2R\mathrm{c}^2}\left((3+\nu)V^2 + \nu(\vec{N}\vec{V})^2 - \frac{\mathrm{G}M}{R}\right). \tag{36}$$

Where the finite mass ratio $\nu = \mu/M$ was introduced. The equations of motion

$$\frac{\mathrm{d}\vec{V}}{\mathrm{d}t} = -\frac{\mathrm{G}M}{R^2}\vec{N} + \frac{\mathrm{G}M}{\mathrm{c}^2R^2}\left(\vec{N}\left[\frac{\mathrm{G}M}{R}(4+2\nu) - V^2(1+3\nu) + \frac{3}{2}\nu(\vec{N}\vec{V})^2\right] + (4-2\nu)(\vec{N}\vec{V})\vec{V}\right) \tag{37}$$

are derived from that Lagrangian using the principle of least action. In the derivation one should notice that $\frac{\mathrm{d}\vec{V}}{dt}$ terms that are accompanied by c^{-2} factors in the formulae can be replaced using the non-relativistic equations of motion (equation (9)).

In order to find the parametrisation of the orbit at 1PN one proceeds analogously to the non-relativistic case. First one uses Noether's theorem to find that the total 1PN energy E and total 1PN angular momentum $\vec{J}$ given by

$$E = \frac{1}{2}V^2 - \frac{\mathrm{G}M}{R} + \frac{3}{8}(1-3\nu)\frac{V^4}{\mathrm{c}^2} + \frac{\mathrm{G}M}{2R\mathrm{c}^2}\left((3+\nu)V^2 + \nu(\vec{N}\vec{V})^2 + \frac{\mathrm{G}M}{R}\right), \tag{38}$$

$$\vec{J} = \vec{R}\wedge\vec{V}\left(1 + \frac{1}{2}(1-3\nu)\frac{V^2}{\mathrm{c}^2} + (3+\nu)\frac{\mathrm{G}M}{R\mathrm{c}^2}\right) \tag{39}$$

are conserved. Introducing polar coordinates and using the relations in equations (12) and (13) one finds after lengthy algebra from equations (38) and (39) that

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = A + \frac{2B}{R} + \frac{C}{R^2} + \frac{D}{R^3}, \tag{40}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{H}{R^2} + \frac{I}{R^3}. \tag{41}$$

I abstain from writing down the coefficients A, B, C, D, H and I explicitly here, but note that they can be expressed using E, J and M. The interested reader may look them up in [6]. It is worth noticing that deriving the relations (40) and (41) requires to drop all terms of order c^{-4} or higher that show up during the derivation. The same is true for the derivation of all of the formulae presented in the following. The derivations also require to use Taylor expansions of denominators in c^{-2} to arrive at the desired results.

When comparing equation (40) with the non-relativistic equation (14) we realise that the only difference is that equation (40) includes an additional R^{-3} dependent term. Performing a transformation of the radial variable as $R = \bar{R} - \frac{D}{2J^2}$ transforms equation (40) to

$$\left(\frac{\mathrm{d}\bar{R}}{\mathrm{d}t}\right)^2 = A + \frac{2B}{\bar{R}} + \frac{\bar{C}}{\bar{R}^2}, \tag{42}$$

where $\bar{C} = C + \frac{BD}{J^2}$. This is now of the same form as equation (14) and therefore the solution is already known. The used transformation of the radial variable as $R' = R + const.$ is known from geometry as a conchoidal transformation. The solution of equation (42) becomes (as mentioned above I only treat the elliptic $A < 0$ case)

$$R = a_R(1 - e_R\cos(u)), \tag{43}$$

where the time dependence of the eccentric anomaly is given as

$$n(t - t_0) = u - e_t\sin(u). \tag{44}$$

The coefficients a_R, e_R, e_t and n can be expressed using only E, J and M. They are listed in [6]. Note that the radial solution in 1PN has the same structure as the Newtonian solution. This is quite surprising and not obvious at all given the enhanced complexity of the problem at 1PN. However, there are two differences. First, one observes that the eccentricity is now split into an angular part e_R and a temporal part e_t. This is an artefact of the conchoidal transformation used to solve the problem: We first parametrised R using equation (43). Afterwards we attempt to solve equation (42). Therefore, we perform the conchoidal transformation which is equivalent to a change of coordinates. In this new coordinate system the parametrisation of R is no longer valid but we need to use $\bar{R} = a_R(1 - e_t\cos(u))$ instead, which leads to the introduction of a second eccentricity in equation (44). Note also that although I use the same notation for the mean motion n here as in the Newtonian solution, it differs from the Newtonian case since 1PN corrections are included now.

Equation (41) can be brought to the same form as its non-relativistic counterpart by using a different conchoidal transformation $R = \tilde{R} + \frac{I}{2H}$:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{H}{\tilde{R}^2} \tag{45}$$

This equation is more difficult to treat than the angular problem. First we introduce a parametrisation for $\tilde{R}$ as $\tilde{R} = \tilde{a}(1 - \tilde{e}\cos(u))$. Note that since we are working in a different coordinate system now we are forced to introduce new parameters $\tilde{a}, \tilde{e}$ which leads to an additional complication. Since the time dependence of the eccentric anomaly is already fixed by equation (44) we arrive at

$$\theta - \theta_0 = \int \frac{H}{n\tilde{a}^2} \frac{1 - e_t\cos(u)}{(1 - \tilde{e}\cos(u))^2} du \tag{46}$$

which can not be reduced as easily as in the non-relativistic case. We solve this problem by introducing yet another eccentricity e_θ. If we define it as $e_\theta \coloneqq 2\tilde{e} - e_t$ we find the relation

$$\frac{1 - e_t\cos(u)}{(1 - \tilde{e}\cos(u))^2} = \frac{1}{1 - e_\theta\cos(u)} \tag{47}$$

which is true up to order c^{-2} since the difference between e_t and $\tilde{e}$ is of the order c^{-2}. The solution of the resulting integral is already knonw and we arrive at

$$\theta - \theta_0 = \frac{H}{n\tilde{a}^2\sqrt{1 - e_\theta^2}} A_{e_\theta} \equiv \frac{\Phi}{2\pi} A_{e_\theta}. \tag{48}$$

The newly introduced coefficients $\tilde{a}$ and e_θ can also be expressed using E, J and M only. At this point we notice a crucial difference to the Newtonian solution in equation (17). In the Newtonian case the prefactor of A_{e_θ} was 1 but now it is different from 1 which leads to a perihelion (or periastron) shift. Therefore, the system moves in a precessing ellipse, as shown in Figure 3. Φ gives the perihelion advance per full orbit. One can also compute the perihelion shift per full orbit at 1PN using $\Delta\theta = \theta(u = 2\pi) - 2\pi$ where $\theta(u = 2\pi)$ is taken from equation (48):

$$\Delta\theta = 6\pi \frac{\mathrm{G}^2 M^2}{J^2 \mathrm{c}^2}. \tag{49}$$

The relative orbital equation analogous to equation (19) is obtained in the same fashion as in the non-relativistic case namely by eliminating u using the radial and angular equations (43) and (48). To do so it is convenient to use yet another conchoidal transformation:

$$R = \frac{e_R}{e_\theta} a_R(1 - e_\theta\cos(u)) + a_R\left(1 - \frac{e_R}{e_\theta}\right) \tag{50}$$

Then the relative orbital equation takes the form

$$R = \left(a_R - \frac{\mathrm{G}\mu}{2\mathrm{c}^2}\right) \frac{1 - e_\theta^2}{1 + e_\theta\cos(A_{e_\theta})} + \frac{\mathrm{G}\mu}{2\mathrm{c}^2}. \tag{51}$$

The only thing left is the calculation of the motions of each point-mass. Using that $V^2 \approx 2\mathrm{G}M/R + 2E$ in equations (34) and (35) as well as the parametrisation for R one finds

$$r = a_r(1 - e_r\cos(u)) \tag{52}$$

where a_r and e_r can be expressed using a_R and e_R. Further, one easily notices that the orbital angle of the first point-mass is equal to the relative angle θ whereas the orbital angle of the second point-mass is equal to $\theta + \pi$ just as in the Newtonian case. Using one last conchoidal transformation

$$r = \frac{e_r}{e_\theta} a_r(1 - e_\theta\cos(u)) + a_r\frac{e_r}{e_\theta} \tag{53}$$

together with equations (52) and (48) we can eliminate u and find

$$r = \left(a_r - \frac{\mathrm{G}m^2m'}{2M^2\mathrm{c}^2}\right) \frac{1 - e_\theta^2}{1 + e_\theta\cos\left((\theta - \theta_0)\frac{\sqrt{J^2 - 6\mathrm{G}^2M^2/\mathrm{c}^2}}{J}\right)} + \frac{\mathrm{G}m^2m'}{2M^2\mathrm{c}^2}. \tag{54}$$

The result for r' can be obtained in the same way. Comparing the orbital equation (54) with the relative orbit equation (51) one notices that both are of the same structure. Since we obtained both equations by conchoidal transformations of the radial parametrisation which governs the equation of a precessing ellipse both orbital equations represent the equation of a conchoid of a precessing ellipse.

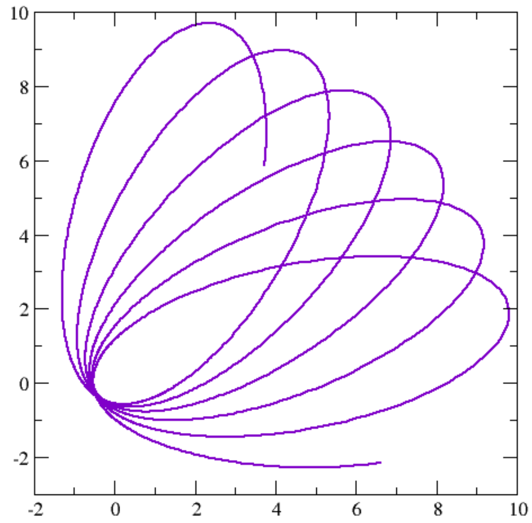

Figure 3: Motion of a particle moving in a precessing ellipse [13].

V The Two-Body Problem at higher orders

The calculations of the equations governing the behaviour of the two-body system taking into account higher order terms become significantly more complicated. The discussion in this chapter follows closely and is partly equivalent with the analysis presented in [9].

V.1 The Two-Body Problem at 2PN

The solution of the two-body problem at 2PN was calculated by Schäfer and Wex [11]. When deriving the equations it has been paid attention to preserving the Keplerian structure of the solution as already done in the derivation at 1PN. For that reason the 1PN solution is sometimes called quasi-Keplerian since it governs the exact same structure but differs by the precession of the perihelion. The solutions obtained at 2PN and also at 3PN posses the same structure of the radial parametrisation but contain additional higher order terms in the angular parametrisation as well as in the equation governing the time dependence of the eccentric anomaly. Therefore, those solutions are sometimes referred to as general quasi-Keplerian.

A further note should be made regarding the coordinate system. One is of course free to choose whatever coordinate system one prefers in order to derive the equations. In the Newtonian and 1PN case harmonic coordinates have been chosen (meaning that each component respects the Laplace equation $\Delta\vec{x} = 0$). However, at 2PN and 3PN it is more convenient to use ADM coordinates named after Arnowitt, Deser and Misner. It can be shown that, starting at 2PN, the derived results in harmonic coordinates start to differ from the results derived in ADM coordinates.

Since the derivation and solution at 2PN is the same as at 3PN except that some additional terms have to be added I do not state the solution nor the derivation here.

V.2 The Two-Body Problem at 3PN

I outline the derivation and the results of the two-body problem at 3PN as derived by R.M. Memmesheimer, A. Gopakumar and G. Schäfer [9], also I do not present any explicit expressions for the coefficients appearing in the formulae. One may look them up in their work. The rather heavy algebra was performed using the software Mathematica [7]. Some computational details are outlined in Appendix B.

Derivation in ADM coordinates The calculation of the solution is done in the Hamiltionian framework instead of the Lagrangian framework as done so far, since the 3PN Hamiltonian was calculated in ADM coordinates before [5]. The equations of motion can be derived using Hamilton's equations $\frac{\mathrm{d}\vec{R}}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial\vec{p}}$ leading to

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \vec{N}\frac{\partial \mathcal{H}}{\partial \vec{p}}, \tag{55}$$

$$R^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = \left|\vec{R}\wedge\frac{\partial \mathcal{H}}{\partial \vec{p}}\right|. \tag{56}$$

Note that in contrast to the relative separation introduced earlier $\vec{R}$ is now scaled by a factor of $\frac{1}{\mathrm{G}M}$ which gets rid of many such factors in the equations. Introducing $s = \frac{1}{R}$ the right-hand sides of the two equations can be represented as seventh degree polynomials in s since the 3PN Hamiltonian contains terms up to the order s^4 (see Appendix B). The major complication with respect to 1PN shows up now. At 1PN we continued by performing a conchoidal transformation to simplify the algebra. This is no longer possible at 3PN.

First we attempt to solve the radial equation (55). We can write the radial equation as

$$\frac{\dot{s}^2}{s^4} = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5 + a_6 s^6 + a_7 s^7. \tag{57}$$

This equation has two non-zero roots s_- and s_+ that stay finite in the limit of $\mathrm{c}\to\infty$. The 3PN eccentricity e_R and semi major axis a_R are defined using the periastron and the perihelion. Since the roots can be related to the maximum and minimum of the radius of the orbit and therefore to the perihelion and the periastron e_R and a_R can be expressed as

$$a_R = \frac{1}{2}\frac{s_- + s_+}{s_- s_+}, \qquad e_R = \frac{s_- - s_+}{s_- + s_+} \tag{58}$$

which allows to calculate 3PN expressions for a_R and e_R (see [9]). To complete the radial solution we also need the 3PN version of the Kepler equation. Therefore, one proceeds by factorising out the two roots and obtains

$$\frac{\dot{s}^2}{s^4} = (b_0 + b_1 s + b_2 s^2 + b_3 s^3 + b_4 s^4 + b_5 s^5)(s - s_-)(s - s_+). \tag{59}$$

Using the Taylor expansion of $(1+x)^{1/2}$ in c^{-1} factors one arrives at

$$t - t_0 = \int_s^{s_-} \frac{A_0 + A_1 s + A_2 s^2 + A_3 s^3 + A_4 s^4 + A_5 s^5}{\sqrt{(s-s_-)(s_+-s)}s^2}\mathrm{d}s. \tag{60}$$

This is an analytically solvable integral (see Appendix B) that can be solved using the radial parametrisation

$$R = a_R(1 - e_R\cos(u)) \tag{61}$$

where the semi major axis and the radial eccentricity are given in equations (58). From this we also obtain an expression for the period P of the system

$$P = 2\int_{s_+}^{s_-} \frac{A_0 + A_1 s + A_2 s^2 + A_3 s^3 + A_4 s^4 + A_5 s^5}{\sqrt{(s-s_-)(s_+-s)}s^2}\mathrm{d}s \tag{62}$$

that is also analytically solvable. The Kepler equation, which completes the radial solution of the problem, relates the mean motion, which is defined as $n = 2\pi/P$ to the eccentric anomaly u. We can use the analytical solutions of equations (60) and (62) to express the Kepler equation in terms of u and $\tilde{v} = 2\arctan\left(\sqrt{\frac{1+e_R}{1-e_R}}\tan(\frac{u}{2})\right)$ for the moment. One finds

$$n(t - t_0) = u + k_0\sin(u) + \frac{k_1}{\mathrm{c}^2}(\tilde{v} - u) + \frac{k_2}{\mathrm{c}^4}\sin(\tilde{v}) + \frac{k_3}{\mathrm{c}^6}\sin(2\tilde{v}) + \frac{k_4}{\mathrm{c}^6}\sin(3\tilde{v}). \tag{63}$$

This is however not the final form of the Kepler equation as we will see shortly.

First, we need to treat the angular equation. The angular equation becomes $\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{\dot{\theta}}{\dot{s}}$ and can be expressed as

$$\theta - \theta_0 = \int_s^{s_-} \frac{C_0 + C_1 s + C_2 s^2 + C_3 s^3 + C_4 s^4 + C_5 s^5}{\sqrt{(s-s_-)(s_+-s)}}\mathrm{d}s \tag{64}$$

by using the roots of equation (57) that were found above. The integral can be calculated analytically as well using again the angular parametrisation given in equation (61) (see Appendix B). Next, one calculates the advance of the perihelion Φ during a full orbit

$$\Phi = 2\int_{s_+}^{s_-} \frac{C_0 + C_1 s + C_2 s^2 + C_3 s^3 + C_4 s^4 + C_5 s^5}{\sqrt{(s-s_-)(s_+-s)}}\mathrm{d}s. \tag{65}$$

Having equations (64) and (65) at hand one can now compute the solution of the angular equation using again the variable $\tilde{v}$

$$\frac{2\pi}{\Phi}(\theta-\theta_0)=\tilde{v}+\frac{d_1}{c^2}\sin(\tilde{v})+\frac{d_2}{c^4}\sin(2\tilde{v})+\frac{d_3}{c^4}\sin(3\tilde{v})+\frac{d_4}{c^6}\sin(4\tilde{v})+\frac{d_5}{c^6}\sin(5\tilde{v}). \tag{66}$$

Note that this version of the angular solution does not reduce to the 1PN version when the higher order terms are neglected due to the $\sin(\tilde{v})$ term. In order to correct for this we need to introduce the angular eccentricity e_θ and the variable $v\equiv A_{e_t}(u)=2\arctan\left(\sqrt{\frac{1+e_\theta}{1-e_\theta}}\tan(\frac{u}{2})\right)$ where e_θ differs from e_R by some PN corrections that are arbitrary at the moment. It is straightforward to replace $\tilde{v}$ by v in equation (66) by using simple trigonometric identities. When doing so the arbitrary part of e_θ is fixed in such a way that the $\sin(\tilde{v})$ term vanishes in order to assure consistency with the 1PN limit. One finds

$$\frac{2\pi}{\Phi}(\theta-\theta_0)=v+\left(\frac{f_{4\theta}}{c^4}+\frac{f_{6\theta}}{c^6}\right)\sin(2v)+\left(\frac{g_{4\theta}}{c^4}+\frac{g_{6\theta}}{c^6}\right)\sin(3v)+\frac{i_{6\theta}}{c^6}\sin(4v)+\frac{h_{6\theta}}{c^6}\sin(5v) \tag{67}$$

for the angular solution.

We are now ready to state the final form of the Kepler equation by replacing $\tilde{v}$ by v in equation (63) and identifying the temporal eccentricity e_t by comparing to the 1PN version. The final form reads

$$n(t-t_0)=u-e_t\sin(u)+\left(\frac{g_{4t}}{c^4}+\frac{g_{6t}}{c^6}\right)(v-u)+\left(\frac{f_{4t}}{c^4}+\frac{f_{6t}}{c^6}\right)\sin(v)+\frac{i_{6t}}{c^6}\sin(2v)+\frac{h_{6t}}{c^6}\sin(3v). \tag{68}$$

Again all the coefficients can be expressed using the 3PN conserved energy and angular momentum.

Derivation in harmonic coordinates In order to make a comparison of the higher order results with the Keplerian or 1PN solution one needs to solve the problem in harmonic coordinates. As a starting point the 3PN expressions for the conserved energy and angular momentum are required. They have been calculated by L. Blanchet and B. R. Iyer [1]. Those expressions contain logarithmic terms which prevent the calculation of the $\frac{dR}{dt}$ and $\frac{d\theta}{dt}$ terms. Luckily, the treatment of the problem in the near-zone regime of the system allows a certain freedom in the choice of the coordinates while still fulfilling the harmonic gauge condition. Performing such a transformation one can remove the logarithmic terms from the expressions. One then proceeds as outlined in the discussion of the 1PN case and obtains expressions for $\dot{R}^2$ and $\dot{\theta}$ which are again both seventh degree polynomials in s. However, they differ from the expressions found in the ADM gauge. From here on one simply follows the same derivation used in the ADM gauge to find the radial and angular solutions as well as the Kepler equation. One arrives at the following solution for the two-body problem in harmonic coordinates:

$$R=a_R(1-e_R\cos(u)), \tag{69}$$

$$\frac{2\pi}{\Phi}(\theta-\theta_0)=A_{e_\theta}(u)+\left(\frac{f_{4\theta}}{c^4}+\frac{f_{6\theta}}{c^6}\right)\sin(2A_{e_\theta}(u))+\left(\frac{g_{4\theta}}{c^4}+\frac{g_{6\theta}}{c^6}\right)\sin(3A_{e_\theta}(u))+$$
$$\frac{i_{6\theta}}{c^6}\sin(4A_{e_\theta}(u))+\frac{h_{6\theta}}{c^6}\sin(5A_{e_\theta}(u)), \tag{70}$$

$$n(t-t_0)=u-e_t\sin(u)+\left(\frac{g_{4t}}{c^4}+\frac{g_{6t}}{c^6}\right)(A_{e_\theta}(u)-u)+\left(\frac{f_{4t}}{c^4}+\frac{f_{6t}}{c^6}\right)\sin(A_{e_\theta}(u))+ \tag{71}$$

$$\frac{i_{6t}}{c^6}\sin(2A_{e_\theta}(u))+\frac{h_{6t}}{c^6}\sin(3A_{e_\theta}(u)). \tag{72}$$

One easily notices that those are the exact same solutions as obtained in ADM coordinates. However, the coefficients appearing in those expressions differ greatly as one can confirm by studying their explicit expressions listed in [9]. Note also that dropping c^{-4} and higher order terms reduces the solution to the known 1PN solution.

Since the computation of those lengthy expressions involves lots of heavy algebra a consistency check for the obtained solutions is highly desirable. It has been shown that using the Hamilton-Jacobi approach the two coefficients n and Φ should be gauge independent [3]. Meaning one should arrive at the same expressions in both ADM and harmonic coordinates. This can actually be verified which constitutes a strong check for the validity of the solutions.

VI Summary

The two-body problem is one of the standard problems in celestial mechanics. Nowadays, its solution is well known (see equations (16) and (17)). The solution describes a non-precessing ellipse.

Using the Post-Newtonian approximation, one can derive solutions of the two-body problem resembling the simple Keplerian solution. This was shown up to third order, meaning the consideration of terms containing c^{-6} factors. Following the same procedure as in the derivation of the Keplerian solution and using some conchoidal transformations to simplify the algebra one finds the solution at 1PN (see equations (43) and (48)). Due to its resemblance to the Keplerian solution this solution is usually dubbed as quasi-Keplerian solution. At 1PN one also discovers an additional physical effect namely a precession of the ellipse.

Starting at 2PN the algebra becomes a lot more tedious which is mainly due to the fact that the trick of performing conchoidal transformations, which simplified the calculations at first order a lot, can no longer be used. The solution at 2PN was calculated in ADM coordinates but not in harmonic coordinates. However, having the solution at third order in harmonic coordinates at hand one can check that the structure of the solution at 2PN stays the same in both gauges and only the coefficients change upon changing the coordinate system.

The algebra becomes even heavier when considering the problem at 3PN. Nevertheless, the problem was solved in the ADM (see equations (61) and (67)) and harmonic gauge (see equations (69) and (70)) and its structure has once again been found to be the same in both gauges but some additional terms were found with respect to the second order solution. The solutions found at 2PN and 3PN look the same as the quasi-Keplerian solution found at 1PN but they involve some additional terms in the angular solution and the time dependence equation. Therefore, those solutions are dubbed generalised quasi-Keplerian solution. One finds that the two quantities Φ and n are gauge independent. This is also predicted by the Hamilton-Jacobi approach and its validity in the found solution constitutes a powerful consistency check which is highly desirable due to the lengthy calculations required to arrive at those solutions.

The third order solution presented here can be used to establish search templates needed for the detection of gravitational waves or for the improvement of the accuracy of the timing formula used for radio observations of relativistic binary pulsars. The steady improve of sensitivity in observational astrophysics will most likely make it necessary for theoreticians to improve the accuracy of their predictions even further in the future.

Appendix A - Derivation of the Newtonian Solution

In order to derive the radial solution one first notices that there are two extrema of R called the periastron (denoted as R_-) and the perihelion (denoted as R_+). One uses the relations

$$\left.\frac{\mathrm{d}R}{\mathrm{d}t}\right|_{R=R_+} = A + \frac{2B}{R_+} + \frac{C}{R_+^2} = 0,$$
$$\left.\frac{\mathrm{d}R}{\mathrm{d}t}\right|_{R=R_-} = A + \frac{2B}{R_-} + \frac{C}{R_-^2} = 0$$

to replace the coefficients B and C by

$$B = -\frac{A}{2}(R_1 + R_-),$$
$$C = AR_-R_+.$$

One continous by parametrising the radial component by

$$R = \frac{1}{2}(R_+ + R_-)(1 - \frac{R_+ - R_-}{R_+ + R_-}\cos(u)) \equiv a(1 - e\cos(u))$$

where I introduced the semi major axis and the eccentricity as

$$a = \frac{1}{2}(R_+ + R_-), \qquad e = \frac{R_+ - R_-}{R_+ + R_-}.$$

Using this we can now write the radial equation (14) as

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = A - \frac{2Aa}{R} + \frac{Aa^2(1-e^2)}{R^2}.$$

In the resulting integral we use the substitution $R = a(1 - e\cos(u))$ and some trigonometry to arrive at

$$t - t_0 = \frac{a}{\sqrt{-A}} \int 1 - e\cos(u)\mathrm{d}u.$$

Noticing that we are in the elliptic case where $A < 0$ this equation is well defined and the solution becomes

$$u - e\sin(u) = \frac{\sqrt{-A}}{a}(t - t_0) \equiv n(t - t_0)$$

which is the Kepler equation we searched for.

To solve the angular equation (17) we start by writing down the integral obtained by using $R = a(1 - e\cos(u))$ in equation (17)

$$\theta - \theta_0 = \int \frac{H}{a^2(1 - e\cos)^2} \frac{\mathrm{d}t}{\mathrm{d}u} \mathrm{d}u$$

By derivating the Kepler equation (18) with respect to time one finds

$$\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{2}{n}(1 - e\cos(u)).$$

Using this and that

$$\frac{\mathrm{d}A_e}{\mathrm{d}u} = \frac{\sqrt{1 - e^2}}{1 - e\cos(u)}$$

one can perform the integration and finds

$$\theta - \theta_0 = \frac{H}{a^2 n\sqrt{1 - e^2}} A_e$$

which is the angular solution.

Appendix B - Some Computational Details

Finding the right hand side expressions of equations (55) and (56)

The computations of the right-hand sides of equations (55) and (56) as seventh degree polynomials involves some computationally expensive steps. Especially it requires to express the radial and angular components of the momentum vector $\vec{p}$ as functions of the 3PN energy E and angular momentum J. In order to reduce the computation time needed for this calculation it is advisable to calculate the results order by order. Therefore, one first solves the problem considering only the Newtonian terms and a Newtonian energy E_0 as well as Newtonian angular momentum J_0. One obtains the Newtonian parts of the momentum vector components p_{r0} and $p_{\theta 0}$. In the next step 1PN terms are considered as well and the equations are expressed in terms of the 1PN energy $E_0 + \frac{1}{c^2}E_1$ and angular momentum $J_0 + \frac{1}{c^2}J_1$. One solves for the 1PN components $p_{r0} + \frac{1}{c^2}p_{r1}$ and $p_{\theta 0} + \frac{1}{c^2}p_{\theta 1}$. Assuming that the Newtonian parts are already known as functions of E_0 and J_0 this can be solved with not too much effort. One proceeds until 3PN order is reached.

Finding the roots of the 7th degree polynomial in equation (57)

Computing the roots of a 7th degree polynomial directly requires a large computational effort since there is no closed expression available for the solution of such an equation. However, using a similar scheme as in the calculations described above the roots can be found without much effort. Considering only Newtonian terms reduces the problem to finding the roots of a quadratic equation. Considering 1PN terms in a second step leads to a 4th degree equation. But writing the roots as $s_+ = s_{+0} + \frac{1}{c^2}s_{+1}$ and $s_- = s_{-0} + \frac{1}{c^2}s_{-1}$ where the Newtonian parts are already known from the first steps reduces the 4th degree equation to two linear equations allowing to easily calculate s_{+1} and s_{-1}. Continouation to 3PN leads to the searched for 3PN roots of the 7th degree polynomial.

Solution of equation (60)

The integral

$$\int \frac{A_0 + A_1 s + A_2 s^2 + A_3 s^3 + A_4 s^4 + A_5 s^5}{\sqrt{(s - s_-)(s_+ - s)s^2}} \mathrm{d}s$$

showing up in equation (60) can be decomposed in five integrals of the form

$$\int \frac{A_x s^x}{\sqrt{(s - s_-)(s_+ - s)s^2}} \mathrm{d}s$$

where x takes integer values between 0 and 5. Using the radial parametrisation given in equation (61) the integral can be reduced to

$$\frac{A_x}{a_R^{x-1}\sqrt{s_+s_-}}\int(1-e_R\cos(u))^{1-x}\mathrm{d}u$$

which can be solved easily by computational software or even by hand.

Solution of equation (64)

Using the radial parametrisation and the exact same procedure as in the calculation of the integral in equation (60) the integral

$$\int\frac{C_0+C_1s+C_2s^2+C_3s^3+C_4s^4+C_5s^5}{\sqrt{(s-s_-)(s_+-s)}}\mathrm{d}s$$

showing up in equation (64) is reduced to integrals of the form

$$\frac{C_x}{a_R^{x+1}\sqrt{s_+s_-}}\int(1-e_R\cos(u))^{-(x+1)}\mathrm{d}u.$$

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