$\mathcal{N} = 4$ Super Yang-Mills Amplitudes: Multi-loop, multi-leg and sometimes multi-Regge

Georgios Papathanasiou

DESY

University of Zurich, November 6, 2017

1606.08807 + in progress w/
Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,Verbeek
1612.08976 w/ Dixon,Drummond,Harrington,McLeod,Spradlin
+ in progress w/ Caron-Huot,Dixon,McLeod,von Hippel

Outline

Aim: Can we compute scattering amplitudes in $SU(N)$ $\mathcal{N} = 4$ super Yang Mills theory to all loops, for any multiplicity and quantum numbers of the external particles?

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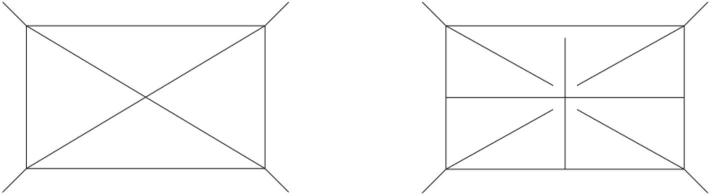

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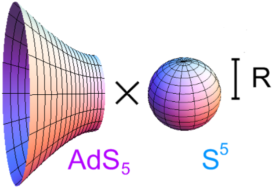

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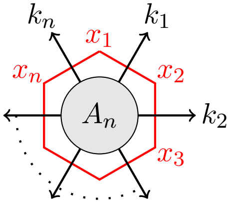

$$k_i = x_{i+1} - x_i$$

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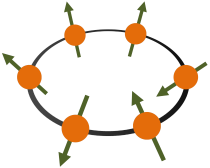

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- Integrable structures $\Rightarrow$ All loop quantities! [Beisert,Eden,Staudacher]

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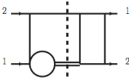
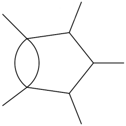


- 2-loop planar all-plus 5-gluon QCD amplitude [Gehrmann,Henn,Lo Presti]



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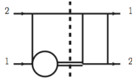

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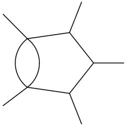

See also recent 3-loop QCD soft anomalous dimension via bootstrap. [Almelid,Duhr,Gardi,McLeod,White]

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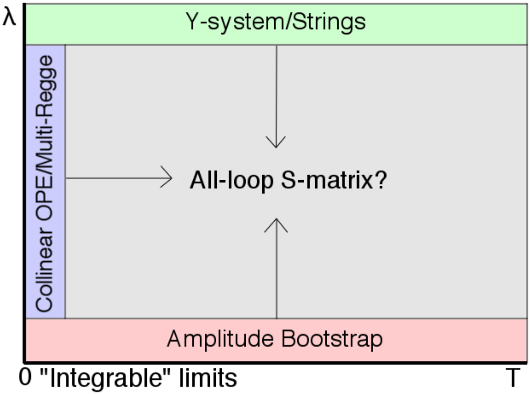
More generally,



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Motivated by this progress, we upgraded this procedure for $n = 7$, with information from the cluster algebra structure of the kinematical space. Surprisingly, more powerful than $n = 6$! [Drummond,GP,Spradlin]

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f_k is a MPL of weight k if its differential may be written as a finite linear combination

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Convenient tool for describing them: The **symbol** $\mathcal{S}(f_k)$ encapsulating recursive application of above definition (on $f_{k-1}^{(\alpha)}$ etc)

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Empeirical evidence: L-loop amplitudes=MPLs of weight $k = 2L$

[Duhr,Del Duca,Smirnov] [Arkani-Hamed,Bourjaily,Cachazo,Goncharov,Postnikov,Trnka] [GP]

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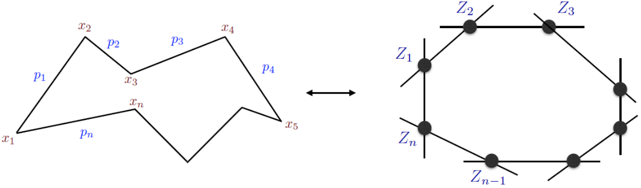
The latter is a collection of n ordered *momentum twistors* Z_i on $\mathbb{P}^3$, (an equivalent way to parametrise massless kinematics), modulo dual conformal transformations. [Hodges]



$$x_i \sim Z_{i-1} \wedge Z_i$$

$$(x_i - x_j)^2 \sim \epsilon_{IJKL} Z_{i-1}^I Z_i^J Z_{j-1}^K Z_j^L = \det(Z_{i-1} Z_i Z_{j-1} Z_j) \equiv \langle i-1 i j-1 j \rangle$$

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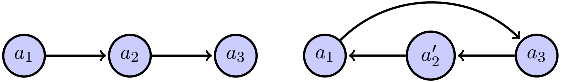

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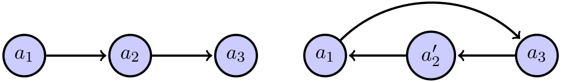

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$a_2' = (a_1 + a_3)/a_2$

and so on...

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2. In new quiver/cluster, $a_k \to a_k' = \Big(\prod_{\text{arrows } i\to k} a_i + \prod_{\text{arrows } k\to j} a_j \Big)/a_k$

Connection to the kinematic space

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See also very interesting, recent work on "cluster adjacency".
[Drummond,Foster,Gürdogan]

Heptagon Symbol Letters

Multiply $\mathcal{A}$-coordinates with suitable powers of $\langle i\, i+1\, i+2\, i+3\rangle$ to form conformally invariant cross-ratios,

$$a_{11} = \frac{\langle 1234\rangle\langle 1567\rangle\langle 2367\rangle}{\langle 1237\rangle\langle 1267\rangle\langle 3456\rangle}\,, \qquad a_{41} = \frac{\langle 2457\rangle\langle 3456\rangle}{\langle 2345\rangle\langle 4567\rangle}\,,$$

$$a_{21} = \frac{\langle 1234\rangle\langle 2567\rangle}{\langle 1267\rangle\langle 2345\rangle}\,, \qquad a_{51} = \frac{\langle 1(23)(45)(67)\rangle}{\langle 1234\rangle\langle 1567\rangle}\,,$$

$$a_{31} = \frac{\langle 1567\rangle\langle 2347\rangle}{\langle 1237\rangle\langle 4567\rangle}\,, \qquad a_{61} = \frac{\langle 1(34)(56)(72)\rangle}{\langle 1234\rangle\langle 1567\rangle}\,,$$

where

$$\langle ijkl\rangle \equiv \langle Z_i Z_j Z_k Z_l\rangle = \det(Z_i Z_j Z_k Z_l)$$

$$\langle a(bc)(de)(fg)\rangle \equiv \langle abde\rangle\langle acfg\rangle - \langle abfg\rangle\langle acde\rangle\,,$$

together with a_{ij} obtained from a_{i1} by cyclically relabeling $Z_m \to Z_{m+j-1}$.

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3. Dual superconformal symmetry ⇒ constrains last symbol entry of amplitudes (MHV 7-pts: a_{2j}, a_{3j}) [Caron-Huot,He]

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$$\sum_{\alpha_1,\ldots,\alpha_k} f_0^{(\alpha_1,\alpha_2,\ldots,\alpha_k)} \underbrace{(\phi_{\alpha_1}\otimes\cdots\otimes\phi_{\alpha_k})}_{\text{omitting } \phi_{\alpha_j}\otimes\phi_{\alpha_{j+1}}} \; d\log\phi_{\alpha_j}\wedge d\log\phi_{\alpha_{j+1}} = 0 \quad \forall j\,.$$

3. Dual superconformal symmetry ⇒ constrains last symbol entry of amplitudes (MHV 7-pts: a_{2j}, a_{3j}) [Caron-Huot,He]
4. Collinear limit: Bern-Dixon-Smirnov ansatz $\mathcal{A}_n^{\text{BDS}}$ contains all IR divergences ⇒ Constraint on $B_n \equiv \mathcal{A}_n/\mathcal{A}_n^{\text{BDS}}$: $\lim_{i+1\parallel i} B_n = B_{n-1}$

Back to contructing and constraining function space

1. Locality: Amplitudes may only have singularities when intermediate particles go on-shell $\Rightarrow$ constrains first symbol entry (7-pts: a_{1j})
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Define ***n*-gon symbol**: A symbol of the corresponding n-gon alphabet, obeying 1 & 2.

Results [Drummond,GP,Spradlin]

Weight k =	1	2	3	4	5	6
Number of heptagon symbols	7	42	237	1288	6763	?
well-defined in the $7 \parallel 6$ limit	3	15	98	646	?	?
which vanish in the $7 \parallel 6$ limit	0	6	72	572	?	?
well-defined for all $i{+}1 \parallel i$	0	0	0	1	?	?
with MHV last entries	0	1	0	2	1	4
with both of the previous two	0	0	0	1	0	1

Table: Heptagon symbols and their properties.

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The symbol of the three-loop seven-particle MHV amplitude is the only weight-6 heptagon symbol which satisfies the last-entry condition and which is finite in the $7 \parallel 6$ collinear limit.

Comparison with the hexagon case

Weight k =	1	2	3	4	5	6
Number of hexagon symbols	3	9	26	75	218	643
well-defined (vanish) in the $6 \parallel 5$ limit	0	2	11	44	155	516
well-defined (vanish) for all $i{+}1 \parallel i$	0	0	2	12	68	307
with MHV last entries	0	3	7	21	62	188
with both of the previous two	0	0	1	4	14	59

Table: Hexagon symbols and their properties.

Surprisingly, heptagon bootstrap more powerful than hexagon one! Fact that $\lim_{7\parallel 6} R_7^{(3)} = R_6^{(3)}$, as well as discrete symmetries such as cyclic $Z_i \to Z_{i+1}$, flip $Z_i \to Z_{n+1-i}$ or parity symmetry **follow for free**, not imposed a priori.

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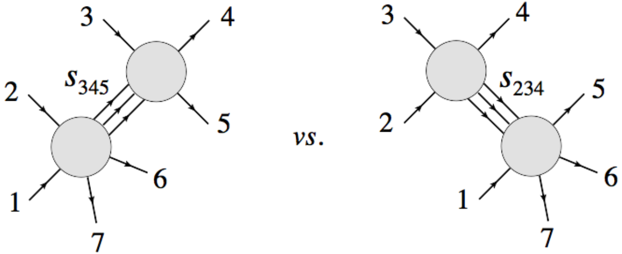

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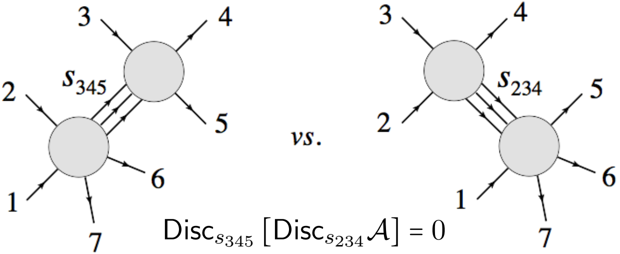

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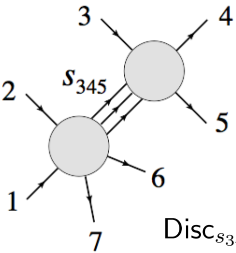

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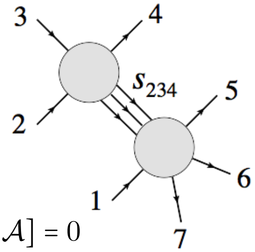

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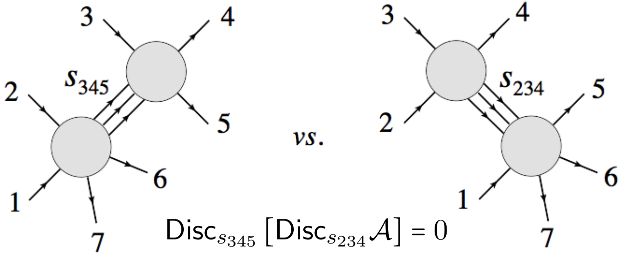

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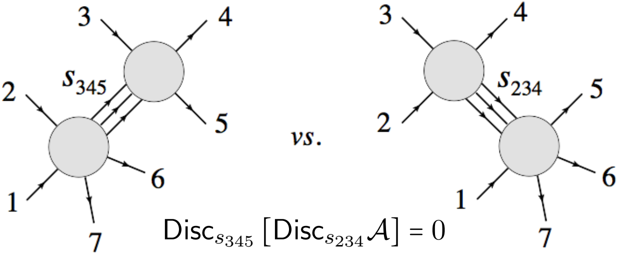

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Heptagon: No $a_{1,i\pm1}, a_{1,i\pm2}$ after $a_{1,i}$ on second symbol entry

Results: Steinmann Heptagon symbols

Weight k =	1	2	3	4	5	6	7	$7''$
parity +, flip +	4	16	48	154	467	1413	4163	3026
parity +, flip −	3	12	43	140	443	1359	4063	2946
parity −, flip +	0	0	3	14	60	210	672	668
parity −, flip −	0	0	3	14	60	210	672	669
Total	7	28	97	322	1030	3192	9570	7309

Table: Number of Steinmann heptagon symbols at weights 1 through 7, and those satisfying the MHV next-to-final entry condition at weight 7. All of them are organized with respect to the discrete symmetries $Z_i \to Z_{i+1}$, $Z_i \to Z_{8-i}$ of the MHV amplitude.

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4. E.g. 6-fold reduction already at weight 5!

In this manner, obtained 3-loop NMHV and 4-loop MHV heptagon

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The 6-loop, 6-particle N+MHV amplitude

[Caron-Huot,Dixon,McLeod,GP,von Hippel;to appear]

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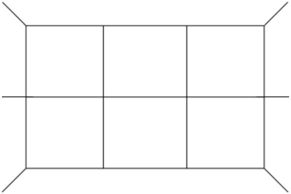

By analyzing its cuts, arguments that following integral, potentially contributing to 6-loop NMHV, is elliptic. [Bourjaily,Parra Martinez]

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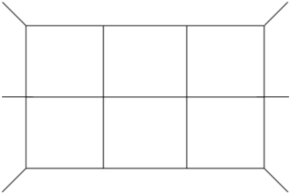

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Our result is purely MPL, thus lending no support to this claim.

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$$a = \frac{u}{vw}, \quad m_u = \frac{1-u}{u}, \quad u = \frac{\langle 6123\rangle\langle 3456\rangle}{\langle 6134\rangle\langle 2356\rangle}, \quad y_u = \frac{\langle 1345\rangle\langle 2456\rangle\langle 1236\rangle}{\langle 1235\rangle\langle 3456\rangle\langle 1246\rangle}$$ & cyclic

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Simplest formulation of Steinmann relations for the amplitude:

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Observed empirically at first, must be consequence of original Steinmann holding not just in the Euclidean region, but also on other Riemann sheets.

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At least for $n = 6$ subspace spanned by double penta-ladder integrals, yes!
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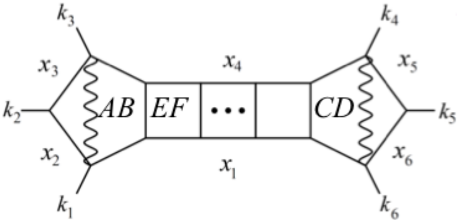

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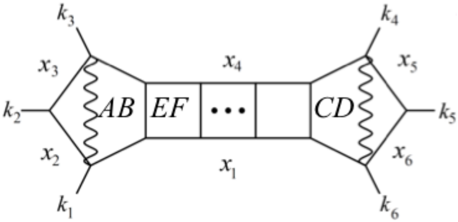

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E.g. $\Omega^{(2)} \equiv \int \frac{d^4 Z_{AB} d^4 Z_{CD} (i\pi^2)^{-2} \langle AB13\rangle\langle CD46\rangle\langle 2345\rangle\langle 5612\rangle\langle 3461\rangle}{\langle AB61\rangle\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle ABCD\rangle\langle CD34\rangle\langle CD45\rangle\langle CD56\rangle\langle CD61\rangle}$

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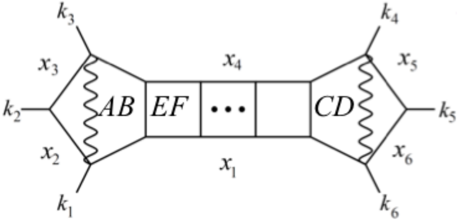

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Can in fact resum $\Omega \equiv \sum \lambda^L \Omega^{(L)}$ in terms of a simple integral.

Beyond seven particles

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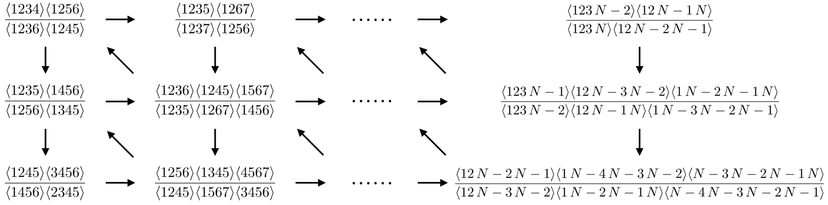
For $N \geq 8$, $Gr(4, N)$ cluster algebra becomes infinite



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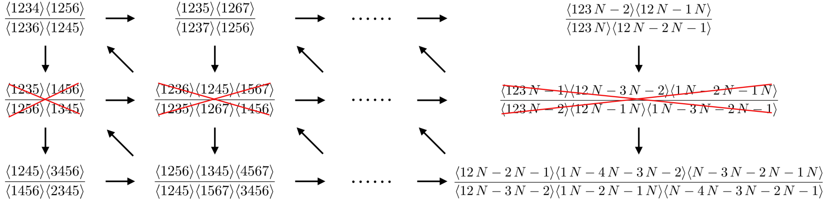

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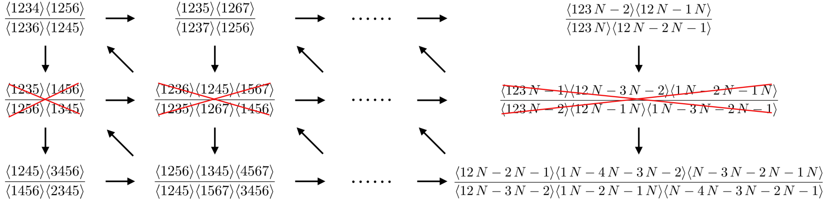

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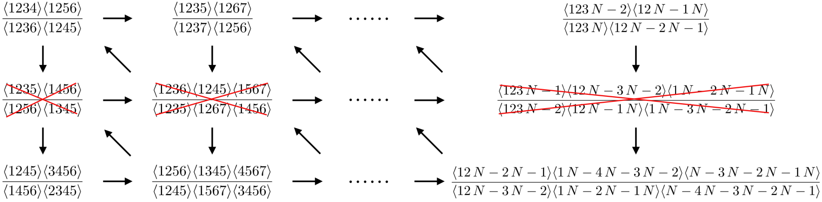

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- Thus, in multi-Regge limit, $Gr(4,N) \to A_{N-5} \times A_{N-5}$: finite!
  [Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,GP,Verbeek]

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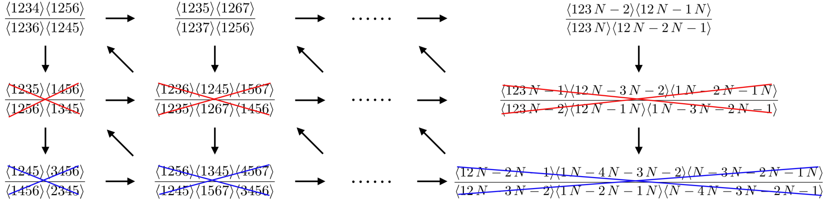

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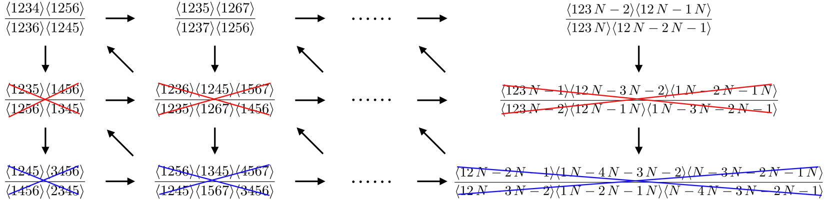
- The two A_{N-5} factors not independent: Related by single-valuedness

Beyond seven particles

For $N \geq 8$, $Gr(4,N)$ cluster algebra becomes infinite

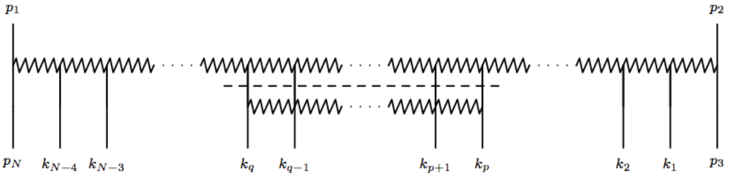


- However, in multi-Regge limit: Middle-row variables $\to 0$, i.e decouple
- Theorem:If quiver has form of Dynkin diagram $\Rightarrow$ finite cluster algebra
- Thus, in multi-Regge limit, $Gr(4,N) \to A_{N-5} \times A_{N-5}$: finite!
  [Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,GP,Verbeek]
- The two A_{N-5} factors not independent: Related by single-valuedness

> Therefore multi-Regge limit crucial for going to higher points.

$2 \to N-2$ Multi-Regge Kinematics (MRK)

Phenomenologically relevant high-energy gluon scattering



Defined by strong ordering of rapidities or lightcone +-components,

$$p_3^+ \gg p_4^+ \gg \dots p_{N-1}^+ \gg p_N^+ \,, \qquad |\mathbf{p}_3| \simeq \dots \simeq |\mathbf{p}_N| \,,$$

where $p^\pm \equiv p^0 \pm p^z, \mathbf{p}_k \equiv p_{k\perp} = p_k^x + i p_k^y$, and can choose $\mathbf{p}_1 = \mathbf{p}_2 = 0$.

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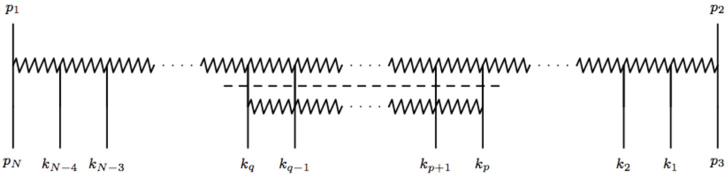

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Implies the hierarchy of scales, for $s_{i\ldots,j} = (p_i + \ldots + p_j)^2$,

$$s_{12} \gg s_{3\cdots N-1}, s_{4\cdots N} \gg s_{3\cdots N-2}, s_{4\cdots N-1}, s_{5\cdots N} \gg \cdots$$

$$\ldots \gg s_{34}, \ldots, s_{N-1N} \gg -s_{23}, \cdots, -s_{2\ldots N} \,.$$

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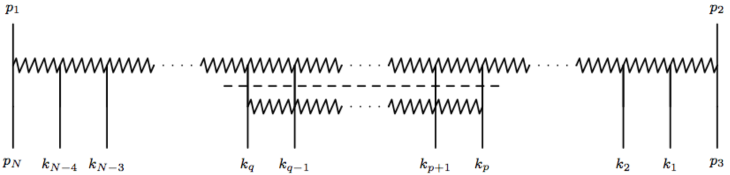

Amplitudes typically develop large logarithms in the kinematic invariants, which are successfully resummed within the Balitsky-Fadin-Kuraev-Lipatov (BFKL) framework, giving rise to the concept of the Reggeized gluon (Regge pole) and its bound states (Regge cuts).

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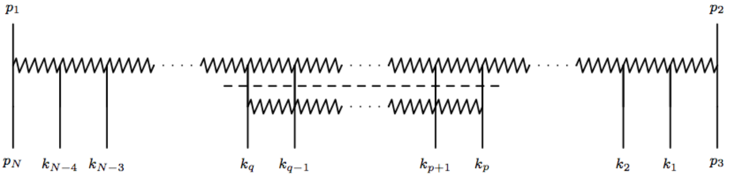

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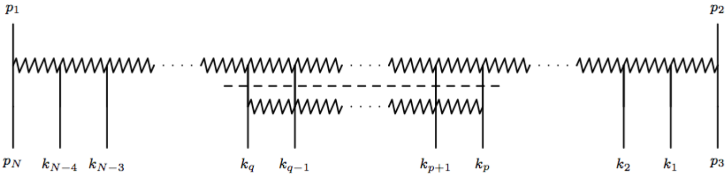

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All $[p, q]$ cuts can be reconstructed from $[1, N-4]$, so focus on the latter.

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> Conclusion: N-particle $\mathcal{N} = 4$ Super Yang-Mills amplitudes in multi-Regge kinematics are described by single-valued A_{N-5} polylogarithms.

Single-valued multiple polylogarithms

Combinations of multiple polylogarithms,

$$G(a_1,\ldots,a_n;z) \equiv \int_0^z \frac{dt_1}{t_1 - a_1} G(a_2,\ldots,a_n;t_1)\,, \quad G(;z) = 1\,,$$

and their complex conjugates, such that all branch cuts cancel, leaving only isolated singularities.

- $\forall\ G(\vec{a},z)$, $\exists$ unique map $\mathbf{s}$, such that $\mathcal{G}(\vec{a},z) \equiv \mathbf{s}\left(G(\vec{a},z)\right)$ is single-valued.
- $G(\vec{a},z)$ then corresponds to *holomorphic* part of $\mathcal{G}(\vec{a},z)$, obtained by setting $\bar{z} \to 0$, and removing any divergent $\log \bar{z}$.

Examples:

$$\mathcal{G}_a(z) = G_a(z) + G_{\bar{a}}(\bar{z}) = \log\left|1 - \frac{z}{a}\right|^2\,,$$

$$\begin{aligned}\mathcal{G}_{a,b}(z) = {} & G_{a,b}(z) + G_{\bar{b},\bar{a}}\left(\bar{z}\right) + G_b(a)G_{\bar{a}}\left(\bar{z}\right) + G_{\bar{b}}\left(\bar{a}\right)G_{\bar{a}}\left(\bar{z}\right) \\ & - G_a(b)G_{\bar{b}}\left(\bar{z}\right) + G_a(z)G_{\bar{b}}\left(\bar{z}\right) - G_{\bar{a}}\left(\bar{b}\right)G_{\bar{b}}\left(\bar{z}\right)\,.\end{aligned}$$

Building on F.Brown's work, we constructed *direct algorithm* for $\mathbf{s}$.

Application: Amplitudes in Leading-logarithmic approximation (LLA)

LLA (Regge cut) contribution factorizes in Fourier-Mellin (FM) space.

- FM transform: $\mathcal{F}[F(\nu,n)] = \sum_{n=-\infty}^{\infty} \left(\frac{w}{\bar{w}}\right)^{n/2} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} |w|^{2i\nu} F(\nu,n)$
- FM maps products into convolutions:

$$\mathcal{F}[F \cdot G] = \mathcal{F}[F] * \mathcal{F}[G] = f * g = \frac{1}{\pi} \int \frac{d^2 w}{|w|^2} f(w)\, g\left(\frac{z}{w}\right)$$

- Implies recursion in loop order. E.g. for N particles: Large logarithms $\prod_{k=1}^{N-5} \log^{i_k} \tau_k$, with $\sum_{i_k} = L-1$ at L loops LLA, and MHV coefficient:

$$g_{+\cdots+}^{(i_1,\ldots,i_k+1,\ldots,i_{N-5})}(w_1,\ldots,w_{N-5}) = \mathcal{E}(w_k) * g_{+\cdots+}^{(i_1,\ldots i_{N-5})}(w_1,\ldots,w_{N-5})$$

In this fashion, obtained LLA contributions of MHV amplitudes to 5 loops for any N, and non-MHV amplitudes up to 4 loops and $N = 8$.

Beyond LLA

Problem: N-particle dispersion integrals diverge for $\log^0 \tau_k$

Explore eikonal approach to 6-particle MRK: [Caron-Huot'13]

$$e^{R_6(w)+i\delta_6(w)} = 2\pi i \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{w}{w^*}\right)^{\frac{n}{2}} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \tilde{\Phi}_n(\nu) |w|^{2i\nu} e^{-L\omega_n(\nu)},$$

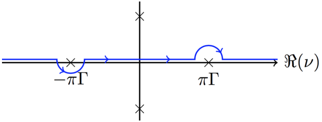
$$L = \log(\tau)+i\pi\,, \quad \delta_6(w) = \pi\Gamma \log\frac{|w|^2}{|1+w|^4}\,, \quad \Gamma = \frac{a}{2} - \frac{\zeta_2}{2}a^2 + \frac{11\zeta_4}{4}a^3 + \mathcal{O}(a^4)\,.$$

Soft limits strongly constrain integrand and integration contour:

$$\lim_{w\to 0} e^{R_6(w)+i\delta_6(w)} = |w|^{2\pi i\Gamma}\,, \quad \lim_{w\to\infty} e^{R_6(w)+i\delta_6(w)} = |w|^{-2\pi i\Gamma}.$$

Imply exact bootstrap conditions for adjoint BFKL eigenvalue ω and impact factor $\tilde{\Phi}$:

$$\omega_0(\pm\pi\Gamma) = 0, \quad \mathrm{Res}_{\nu=\pm\pi\Gamma}\left(\tilde{\Phi}_0(\nu)\right) = \pm\frac{1}{2\pi},$$



Beyond LLA

Heptagon all-loop dispersion relation

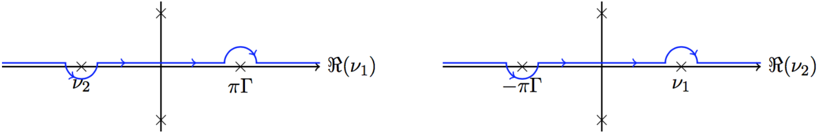
Propose (new ingredient: central emission block $\tilde{C}_{n_1 n_2}$)

$$e^{R_7+i\delta_7} = 2\pi i \sum_{n_1,n_2=-\infty}^{\infty} (-1)^{n_1+n_2} \left(\frac{w_1}{w_1^*}\right)^{\frac{n_1}{2}} \left(\frac{w_2}{w_2^*}\right)^{\frac{n_2}{2}} \int \frac{d\nu_1 d\nu_2}{(2\pi)^2} |w_1|^{2i\nu_1} |w_2|^{2i\nu_2}$$
$$\times\, e^{-L_1\omega_{n_1}(\nu_1)} e^{-L_2\omega_{n_2}(\nu_2)} \tilde{\Phi}_{n_1}(\nu_1) \tilde{C}_{n_1 n_2}(\nu_1,\nu_2) \tilde{\Phi}_{n_2}(\nu_2)\,,$$

$$L_i = \log\tau_i + i\pi\,, \qquad \delta_7 = \pi\Gamma \log\frac{|w_1 w_2|^2}{|1+w_2+w_1w_2|^4}\,.$$

Similarly, soft limits $w_1 \to 0, w_2 \to \infty$ and $w_2 \to 0$ with $w_1 w_2$ fixed, imply

$$\tilde{C}_{0 n_2}(\pi\Gamma,\nu_2) = \tilde{C}_{n_1 0}(\nu_1,-\pi\Gamma) = 2\pi i\,, \quad \operatorname*{Res}_{\nu_1=\nu_2} \tilde{C}_{n_2 n_2}(\nu_1,\nu_2) = \frac{-i(-1)^{n_2} e^{i\pi\omega_{n_2}(\nu_2)}}{\tilde{\Phi}_{n_2}(\nu_2)}$$



Determining the building blocks of the BFKL dispersion integrals $\omega_n, \tilde{\Phi}_n$

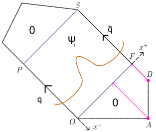
- Initially obtained to LO from adjoint BFKL equation
  [Bartels,Lipatov,Sabio Vera]

$$\omega_n(\nu) = -aE(\nu,n) + \mathcal{O}(a^2)\,, \quad \tilde{\Phi}_n(\nu) = \frac{a}{2}\frac{1}{\nu^2+\frac{n^2}{4}} + \mathcal{O}(a^2)\,,$$

$$E(\nu,n) = -\frac{1}{2}\frac{|n|}{\nu^2+\frac{n^2}{4}} + \psi\left(1+i\nu+\frac{|n|}{2}\right) + \psi\left(1-i\nu+\frac{|n|}{2}\right) - 2\psi(1)\,.$$

- Higher order corrections extracted from 6-particle perturbative data
  [Lipatov,Prygarin] [Dixon,Duhr,Pennington]
- Remarkably, MRK intimately related to collinear limit, described at any coupling with the help of integrability by the 'Wilson loop OPE'
  [Alday,Gaiotto,Maldacena,Sever,Vieira] [Basso,Sever,Vieira]
- Can obtain $\omega_n, \tilde{\Phi}_n$ to all loops! From analytic continuation of '1-particle gluon bound states' [Basso,Caron-Huot,Sever] [Drummond,GP] [Hatsuda]

$$\mathcal{W}_6 \equiv \sum_{a=1}^{\infty} \int \frac{du}{2\pi}\, \mu_a(u) e^{-E_a(u)\tau + ip_a(u)\sigma + ia\phi}$$



Determining the building blocks of the BFKL dispersion integrals $\tilde{C}_{n_1 n_2}$

- Once again, computed to LO within the BFKL approach [Bartels,Kormilitzin, Lipatov, Prygarin]

$$\tilde{C}^{(0)}_{n_1 n_2}(\nu_1,\nu_2) = \frac{\Gamma\left(1-i\nu_1-\frac{n_1}{2}\right)\Gamma\left(1+i\nu_2+\frac{n_2}{2}\right)\Gamma\left(i\nu_1-i\nu_2-\frac{n_1}{2}+\frac{n_2}{2}\right)}{\Gamma\left(i\nu_1-\frac{n_1}{2}\right)\Gamma\left(-i\nu_2+\frac{n_2}{2}\right)\Gamma\left(1-i\nu_1+i\nu_2-\frac{n_1}{2}+\frac{n_2}{2}\right)}$$

- Here: Extract from 2-loop symbol of all MHV amplitudes, specialized to MRK [Caron-Huot] [Prygarin,Spradlin,VerguVolovich] [Barheer,GP,Schomerus]
- Single-valuedness and soft limits *uniquely* upgrade symbol to function:

$$\frac{\tilde{C}^{(1)}_{n_1 n_2}(\nu_1,\nu_2)}{\tilde{C}^{(0)}_{n_1 n_2}(\nu_1,\nu_2)} = \frac{1}{2}\Big[DE_1 - DE_2 + E_1E_2 + \tfrac{1}{4}(N_1+N_2)^2 + V_1V_2 + (V_1-V_2)\big(M-E_1-E_2\big) + 2\zeta_2 + i\pi(V_2-V_1-E_1-E_2)\Big]\,.$$

$$V(\nu,n) \equiv \frac{i\nu}{\nu^2+\frac{n^2}{4}}\,, \qquad N(\nu,n) = \frac{n}{\nu^2+\frac{n^2}{4}}\,, \qquad D_\nu = -i\partial/\partial\nu\,,$$

$$M(\nu_1,n_1,\nu_2,n_2) = \psi\big(i(\nu_1-\nu_2)-\tfrac{n_1-n_2}{2}\big) + \psi\big(1-i(\nu_1-\nu_2)-\tfrac{n_1-n_2}{2}\big) + 2\gamma_E\,.$$

Applications

- 5-loop MHV/4-loop NMHV 7-particle amplitude to NLLA, by evaluating dispersion integral by residues + nested sum algorithms [Moch,Uwer,Weinzierl]
- Generalize dispersion integral to any number of particles! 3-loop MHV 8-particle amplitude to NLLA by convolutions
- Momentun space factorization: L-loop NLLA MHV amplitudes decomposed into building blocks associated to amplitudes with up to $L+5$ legs
- Thus, obtain all 3-loop NLLA MHV amplitudes

Conclusions & Outlook

In this presentation, we talked the beauty and simplicity of $\mathcal{N}=4$ SYM amplitudes.

We focused on two approaches for their computation:

- The (Steinmann, Cluster) Bootstrap at fixed-order/general kinematics, exploiting their analytic properties

  $\Rightarrow$ $N=6$ gluons to 6 loops, $N=7$ gluons to 4 loops

- The study of the multi-Regge limit, where factorization, dual conformal invariance and soft limits yield all-loop predictions $\forall N$

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> Ultimately, can the integrability of planar SYM theory, together with a thorough knowledge of the analytic structure of its amplitudes, lead us to the theory's exact S-matrix?

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$$(x-x')^2 \propto 2X\cdot X' = \epsilon_{IJKL} Z^I \tilde{Z}^J Z'^K \tilde{Z}'^L = \det(Z\tilde{Z}Z'\tilde{Z}') \equiv \langle Z\tilde{Z}Z'\tilde{Z}'\rangle$$
- $(x_{i+i} - x_i)^2 = 0 \quad \Rightarrow X_i = Z_{i-1} \wedge Z_i$

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Comparing the two matrices,

$$\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4,n)/(C^*)^{n-1}$$

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Define a **heptagon symbol**: An integrable symbol with alphabet a_{ij} that obeys first-entry condition.

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Particularly here: Only the 14 letters a_{2j} and a_{3j} may appear in the last symbol entry of R_7.

Imposing Constraints: The Collinear Limit

It is baked into the definition of the BDS normalized n-particle L-loop MHV remainder function that it should smoothly approach the corresponding $(n{-}1)$-particle function in any simple collinear limit:

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> A function has a well-defined $i+1 \parallel i$ limit only if its symbol is independent of all nine of these letters.

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"Just" linear algebra, however for e.g. 4-loop MHV hexagon A boils down to a size of 941498×60182. Tackled with fraction-free variants of Gaussian elimination that bound the size of intermediate expressions, implemented in `Integer Matrix Library` and `Sage`. [Storjohann]

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BDS-like normalized amplitudes obey Steinmann relations, BDS normalized ones do not!

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- $(67) = (76) \equiv [12345]$ *Dual superconformal* R-invariants, with

$$[abcde] = \frac{\delta^{0|4}\big(\chi_a\langle bcde\rangle + \text{cyclic}\big)}{\langle abcd\rangle\langle bcde\rangle\langle cdea\rangle\langle deab\rangle\langle eabc\rangle}\,, \quad \chi_i^A = \sum_{j=1}^{i-1}\langle ji\rangle\eta_j^A\,.$$

NMHV final entry conditions

[Caron-Huot]

$$(34)\log a_{21},\quad (14)\log a_{21},\quad (15)\log a_{21},\quad (16)\log a_{21},\quad (13)\log a_{21},\quad (12)\log a_{21},$$
$$(45)\log a_{37},\quad (47)\log a_{37},\quad (37)\log a_{37},\quad (27)\log a_{37},\quad (57)\log a_{37},\quad (67)\log a_{37},$$
$$(45)\log\frac{a_{34}}{a_{11}},\quad (14)\log\frac{a_{34}}{a_{11}},\quad (14)\log\frac{a_{11}a_{24}}{a_{46}},\quad (14)\log\frac{a_{14}a_{31}}{a_{34}},$$
$$(24)\log\frac{a_{44}}{a_{42}},\qquad (56)\log a_{57},\qquad (12)\log a_{57},\qquad (16)\log\frac{a_{67}}{a_{26}},$$
$$(13)\log\frac{a_{41}}{a_{26}a_{33}}+((14)-(15))\log a_{26}-(17)\log a_{26}a_{37}+(45)\log\frac{a_{22}}{a_{34}a_{35}}-(34)\log a_{33}\,,$$

Results: 3-loop NMHV Heptagon

Loop order L =	1	2	3
Steinmann symbols	15×28	$15{\times}322$	$15{\times}3192$
NMHV final entry	42	85	226
Dihedral symmetry	5	11	31
Well-defined collinear	0	0	0

Results: 3-loop NMHV Heptagon

Loop order L =	1	2	3
Steinmann symbols	15×28	$15{\times}322$	$15{\times}3192$
NMHV final entry	42	85	226
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4. We also need collinear limit of R-invariants

Results: 4-loop MHV Heptagon

Loop order L =	1	2	3	4
Steinmann symbols	28	322	3192	?
MHV final entry	1	1	2	4
Well-defined collinear	0	0	0	0

Results: 4-loop MHV Heptagon

Loop order L =	1	2	3	4
Steinmann symbols	28	322	3192	?
MHV final entry	1	1	2	4
Well-defined collinear	0	0	0	0

For last step, we need to convert BDS-like normalized amplitude F to BDS normalized one $\mathcal{F}$,

$$\mathcal{F} = F e^{\frac{\Gamma_{\text{cusp}}}{4} Y_7} \xrightarrow[\Gamma_{\text{cusp}} \to 4g^2]{\text{symbol}} \mathcal{F}^{(L)} = \sum_{k=0}^{L} F^{(k)} \frac{Y_n^{L-k}}{(L-k)!} \, .$$

Results: 4-loop MHV Heptagon

Loop order L =	1	2	3	4
Steinmann symbols	28	322	3192	?
MHV final entry	1	1	2	4
Well-defined collinear	0	0	0	0

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Independence of $\lim_{i+1 \| i} \mathcal{F}$ on 9 additional letters no longer a homogeneous constraint, fixes amplitude completely!

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Steinmann symbols	28	322	3192	?
MHV final entry	1	1	2	4
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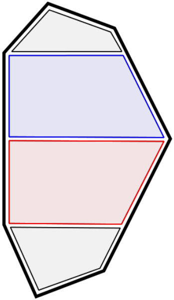
Strong tension between collinear properties and Steinmann relations.

Further check: Heptagon Wilson loop OPE

This is an expansion in two variables $e^{-\tau_1}, e^{-\tau_2}$ near the double collinear limit $\tau_1 \to \infty, \tau_2 \to \infty$.



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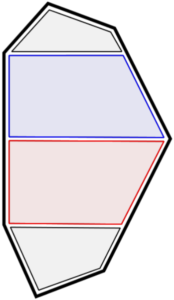

Integrability predicts linear terms in $e^{-\tau_i}$ to all loops in integral form, e.g.[Basso,Sever,Vieira 2]

$$h = e^{i(\phi_1+\phi_2)}\, e^{-\tau_1-\tau_2} \int \frac{du dv}{(2\pi)^2}\, \mu(u)\, P_{FF}(-u|v)\mu(v) \times$$
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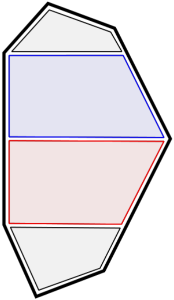

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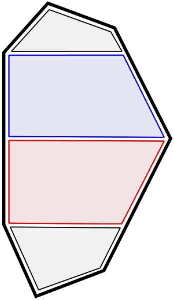

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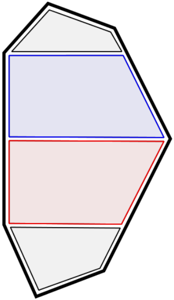

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Perfect match, currently computing 4 loops

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