Large-n_f Contributions to the Four-Loop Splitting Functions in QCD

Nucl. Phys. B915 (2017) 335-362 [arXiv:1610.07477],
[arXiv:1711.05267]

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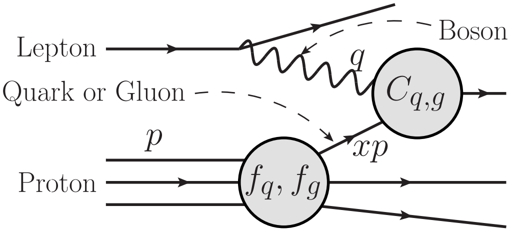
Theoretical Particle Physics Seminar, Zurich, 28/11/2017

INTRODUCTION

Deep Inelastic Scattering: a lepton scatters from a proton.



Boson: γ, H, Z^0 (Neutral Current) or $W^{\pm}$ (Charged Current)

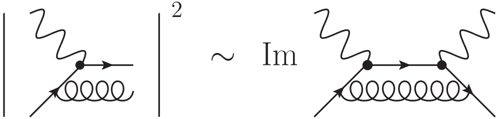
Cross-section: $\sigma \sim \sum_a F_a(x, Q^2 = -q^2 > 0) = \sum_a \left[C_{a,q} \otimes f_q + C_{a,g} \otimes f_g\right]$

x – *"Collinear momentum fraction"*
F_a – *"Structure Function"*
$C_{a,j}$ – *"Coefficient Function"*
$\otimes$ – *"Mellin Convolution"*
f_j – *"Parton Distribution Function"*
$a = 2, 3, L, \phi$

INCLUSIVE DIS

Sum over final states. To compute $C_{a,q}, C_{a,g}$, we use the **optical theorem**.

Compute **forward scattering amplitudes**:



Use Dim. Reg. ($D = 4 - 2\varepsilon$). Divergences appear as poles in ε.

Renormalization of a_s removes UV poles. "Collinear" poles remain,

$$\tilde{C}_{a,j} = \tilde{C}_{a,j}\left(x, a_s(\mu_r^2), Q^2/\mu_r^2, \varepsilon\right) .$$

Collinear Factorization

We need to deal with these collinear poles: renormalize the PDF.

$$F_a = \tilde{C}_{a,j} \otimes \tilde{f}_j = C_{a,j} \otimes Z_{ji}\left(x, a_{\mathrm{s}}, {}^{\mu_{\mathrm{r}}^2}/_{\mu_{\mathrm{f}}^2}, \varepsilon\right) \otimes \tilde{f}_i = C_{a,j} \otimes f_j.$$

Factorize $\tilde{C}_{a,j}$: $C_{a,j}$ is finite. Z_{ji} contains only poles in ε.

Factorization at scale μ_{f}^2, implies f_j **has scale dependence**:

$$\frac{d}{d\ln\mu_{\mathrm{f}}^2} f_j = \frac{d}{d\ln\mu_{\mathrm{f}}^2} Z_{ji} \otimes \tilde{f}_i = \underbrace{\frac{d}{d\ln\mu_{\mathrm{f}}^2} Z_{jk} \otimes Z_{ki}^{-1}}_{P_{ji}} \otimes f_i.$$

- this is the DGLAP evolution equation
- P_{ji} are the **Splitting Functions**

Know Z_{ji} from calculation of $\tilde{C}_{a,j}$, so we can extract P_{ji}.

PDFs are universal to all hadron interactions; Splitting Functions are also.

SPLITTING FUNCTIONS

DGLAP evolution: system of $2n_f$+1 coupled equations.

By defining the distributions

$$q_s = \sum_{i=1}^{n_f}(f_i + \bar{f}_i), \qquad q^{\pm}_{ns,ij} = (f_i \pm \bar{f}_i) - (f_j \pm \bar{f}_j), \qquad q_V = \sum_{i=1}^{n_f}(f_i - \bar{f}_i),$$

we have the evolution equations, (setting $\mu_{\mathrm{f}}^2 = Q^2$):

$$\frac{d}{d\ln Q^2}\begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix},$$

$$\frac{d}{d\ln Q^2} q^{\pm}_{ns,ij} = P^{\pm}_{ns} \otimes q^{\pm}_{ns,ij}, \qquad \frac{d}{d\ln Q^2} q_V = P_V \otimes q_V.$$

$$\boxed{P_{ij},\ P^{\pm}_{ns},\ P_V}$$

In Mellin space...

Take the **Mellin transform**; convolutions ($\otimes$) become products.

$$F_a(N, Q^2) = \int_0^1 \mathrm{d}x \; x^{N-1} \hat{F}_a(x, Q^2).$$

We compute **Mellin moments** of $\tilde{C}_{a,j}$, $N = 2, 4, 6, \ldots$, **not** an analytic expression in N (which gives x-space expression via Inverse MT).

- Projection operator:

$$\mathcal{P}_N = \frac{q^{\{\mu_1} \cdots q^{\mu_N\}}}{N!} \frac{\partial^N}{\partial p^{\mu_1} \cdots \partial p^{\mu_N}} \bigg|_{p=0}$$

- Mellin moments of $C_{a,j}$ and P_{ij}.

Q: Given some fixed number of Mellin moments of P_{ij}, can we derive an analytic expression for general N?

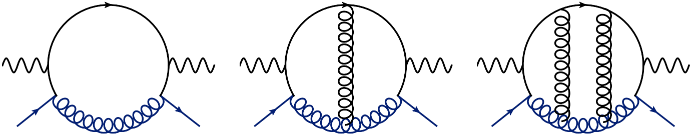
- **this is the goal here.**

SOFTWARE

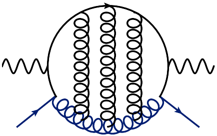
qgraf: generate diagrams [Nogueira '93]

TFORM 4.2: physics, project Mellin moments. [Ruijl,Ueda,Vermaseren '17]
Produces 2-point integrals. Need to reduce to masters...

To 3 loops, we can use **MINCER**. [Larin,Tkachov,Vermaseren '91]



At 4 loops, **FORCER**. [Ruijl,Ueda,Vermaseren '17]



What do P_{ij} "look like"?

To a_s^3 (3 loops), written in terms of **harmonic sums**,

$$S_m(N) = \sum_{i=1}^{N} \frac{1}{i^m}, \quad S_{-m}(N) = \sum_{i=1}^{N} \frac{(-1)^i}{i^m},$$

$$S_{[-]m_1,m_2,\ldots,m_l}(N) = \sum_{i=1}^{N} \frac{[(-1)^i]}{i^{m_1}} S_{m_2,\ldots,m_l}(i),$$

and **denominators**, $D_i^p = \left(\frac{1}{N+i}\right)^p$.

Define

- **harmonic weight**: $\sum_{i=1}^{l} |m_i|$,
- **overall weight**: harmonic weight + p.

$$P_{ij} = \sum_{n=0}^{\infty} a_s^{n+1} P_{ij}^{(n)},$$

to a_s^3, $P_{ij}^{(n)}$ written as terms of overall weight up to $(2n+1)$.

2-Loop Example

$$P_{qg}^{(1)}\Big|_{C_A n_f}(N) = -\Big[8(2D_2 - 2D_1 + D_0)\mathbf{S}_{-2} + 8(2D_2 - 2D_1 + D_0)\mathbf{S}_{1,1} + 16(D_2^2 - D_1^2)\mathbf{S}_1 + 8(4D_2^3 + 2D_1^3 + D_0^3)\Big]_{OW3} - \Big[\frac{4}{3}(44D_2^2 + 12D_1^2 + 3D_0^2)\Big]_{OW2} + \Big[\frac{4}{9}(20D_{-1} - 146D_2 + 153D_1 - 18D_0)\Big]_{OW1}$$

- At overall weight i, up to factor $(1/3)^{(3-i)}$, coefficients are **integers**.

Possible basis:

$$\{S_{-2}, S_{1,1}, S_2\} \cdot \{D_0, D_1, D_2\}$$
$$\{S_1\} \cdot \{D_0^{1,2}, D_1^{1,2}, D_2^{1,2}\}$$
$$\{1\} \cdot \{D_0^{1,2,3}, D_1^{1,2,3}, D_2^{1,2,3}, D_{-1}\},$$

Need to determine **25 coefficients**.

2-Loop Example

Compute Mellin moments:

$$
\begin{aligned}
P_{qg}^{(1)}\Big|_{C_A n_f}(2) &= \quad 35/27 \\
P_{qg}^{(1)}\Big|_{C_A n_f}(4) &= -\,16387/9000 \\
P_{qg}^{(1)}\Big|_{C_A n_f}(6) &= -\,867311/370440 \\
P_{qg}^{(1)}\Big|_{C_A n_f}(8) &= -\,100911011/40824000 \\
&\vdots
\end{aligned}
$$

With moments $N = 2, 4, \ldots, 50$ we can solve for the 25 basis coefficients.

Can we do better?

- Assume $(1/3)^{(3-i)}$, produce a basis with **integer** coefficients,
- We have a **system of Diophantine equations –** solve!

Lattice Basis Reduction

Lenstra-Lenstra-Lovász Lattice Basis Reduction: [Lenstra,Lenstra,Lovász '82]

- find a short lattice basis in polynomial time
- can be used to find integer solutions to equations

`axb`:

- part of **`calc`** [`www.numbertheory.org`]
- LLL-based solver for **systems of Diophantine equations**

See also, `Mathematica`, `Maple`, `fpLLL`, ... , many more.

To solve (two-loop example):

$$\begin{pmatrix} b_1(2), \; \ldots \;, \; b_{25}(2) \\ \vdots \\ b_1(m), \; \ldots \;, \; b_{25}(m) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_{25} \end{pmatrix} = \begin{pmatrix} P_{qg}^{(1)}\big|_{C_A n_f}(2) \\ \vdots \\ P_{qg}^{(1)}\big|_{C_A n_f}(m) \end{pmatrix}$$

$b_i(N)$: basis elements. $c_i \in \mathbb{Z}$: coefficients.

Simple LLL Example

Suppose $r = 1.61803$ is a (rounded) solution to a quadratic equation.

Form the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 100\,000r^2 \\ 0 & 1 & 0 & 100\,000r \\ 0 & 0 & 1 & 100\,000 \end{pmatrix}.$$

A new basis consists of vectors of the form $(a, b, c, 100\,000(ar^2 + br + c))$.

We seek **short** vectors. In particular, $100\,000(ar^2 + br + c)$ must be small.

Apply `LatticeReduce[]` (`Mathematica`):

$$\begin{pmatrix} -1 & 1 & 1 & 1 \\ 114 & -258 & 119 & 254 \\ -103 & 14 & 247 & -364 \end{pmatrix}.$$

$$-r^2 + r + 1 = 0 \implies \boxed{r = 1.61803\ldots \checkmark}$$

$$114r^2 - 258r + 119 = 0 \implies r = 1.61801\ldots$$

$$-103r^2 + 14r + 247 = 0 \implies r = 1.61802\ldots$$

2-Loop Example: Reconstruction

Determines $P_{qg}^{(1)}\Big|_{C_A n_f}$ (**25** integer coefficients) with **just 9** Mellin moments.

- **`axb`** solution, $(c_1, \ldots, c_{25}) =$

$$(\underbrace{2, 6, 72, 8, 88, 584, 4, 24, -612, -80}_{SW0}, \underbrace{0, 0, 4, 0, -4, 0}_{SW1}, \underbrace{2, 4, -4, 2, 4, -4, 0, 0, 0}_{SW2})$$

What if the basis were incorrect? For e.g., leave out D_{-1}:

- solve with $N = 2, \ldots, 18$,

$$(-43, 423, 123, 1492, -102, 1332, 4, 24, -612, -15, 437, 102, -2399, 80, 1700, -146, 180, -26, -1065, 670, 579, -919, 490, 605)$$

- solve with $N = 2, \ldots, 20$,

$$(-178, 4391, -25712, 412, -10348, -6476, 4, 24, -612, -572, 25401, -2178, -5642, -3526, -20152, -3302, -3161, 6474, -4011, 5092, 3775, -3283, -4617, 11029)$$

Claim: these solutions are **"obviously bad"**.

Four-Loop Splitting Functions

Large-n_f contributions:

- subset of diagrams, much easier for **`FORCER`** to compute (insertions)
- smaller reconstruction bases (terms of lower overall weight)

Singlet Splitting Functions, colour factors at n_f^3,

$$P_{qq}^{(3)}\{C_F n_f^3\} \quad P_{qg}^{(3)}\{C_A n_f^3, C_F n_f^3\}$$
$$P_{gq}^{(3)}\{C_F n_f^3\} \quad P_{gg}^{(3)}\{C_A n_f^3, C_F n_f^3\}$$

Guess bases using lower order information. Number of coefficients:

$$P_{qq}^{(3)}\{69\} \quad P_{qg}^{(3)}\{125, 101\}$$
$$P_{gq}^{(3)}\{38\} \quad P_{gg}^{(3)}\{34, 54\}$$

Moments used for reconstruction, (check), $N = 2, 4, \ldots$

$$P_{qq}^{(3)}\{30(44)\} \quad P_{qg}^{(3)}\{\times(\times), 40(54)\}$$
$$P_{gq}^{(3)}\{18(28)\} \quad P_{gg}^{(3)}\{20(28), 26(32)\}$$

Guessing a Basis

Crucial that the basis is "just right". Too small/big: no (good) solution.

$$P^{(1)}_{qg}\big|_{C_A n_f^1}\ [OW3], \qquad P^{(2)}_{qg}\big|_{C_A n_f^2}\ [OW4], \qquad P^{(3)}_{qg}\big|_{C_A n_f^3}\ [OW5]\ ?$$

Look at moments: $\mathbf{1/13^5}$ can **only** come from D_1^5:

$$P^{(3)}_{qg}\big|_{C_A n_f^3}(N=12) = \frac{894866035734231246739}{2^3\, 3^{10}\, 5^4\, 7^5\, 11^3\, 13^5}.$$

H. Sums	Denominators		
$SW4$			ρ
$SW3$		ρ	$D_0^2, D_1^{1,2}, D_2^{1,2}, D_{-1}$
$SW2$	ρ	$D_1^{1,2}, D_2^{1,2}$	$D_0^{2,3}, D_1^3, D_2^3, D_{-1}$
$SW1$	$D_0^{1,2}, D_1^{1,2}, D_2^{1,2}$	D_1^3, D_2^3	$D_0^{3,4}, D_1^4, D_2^4, D_{-1}$
$SW0$	$D_0^{1,2,3}, D_1^{1,2,3}, D_2^{1,2,3}, D_{-1}$	D_0^4, D_1^4, D_2^4	D_0^5, D_1^5, D_2^5

SWN: "weight N harmonic sums, no index -1". $\quad \rho = D_0 - 2D_1 + 2D_2$.

sums in sets $SW\{0,1,2,3,4,5\} : \{1,1,3,7,17,41\}$.

Guessing a Basis

We need to provide sufficient powers of $\mathbf{1/3}$ at each overall weight.

OW	5	4	3	2	1
$P^{(1)}_{qg}\big\|_{C_A n_f^1}$			1	$1/3$	$1/9$
$P^{(2)}_{qg}\big\|_{C_A n_f^2}$		$1/3$	$1/9$	$1/27$	$1/81$
$P^{(3)}_{qg}\big\|_{C_A n_f^3}$	$1/9?$		$\cdots$		$1/729?$

Look at moments:

$$P^{(3)}_{qg}\big|_{C_A n_f^3}(N=8) = \frac{886247558029}{3^{13}\,5^5\,7^3}$$

$$P^{(3)}_{qg}\big|_{C_A n_f^3}(N=26) = \frac{4099414476820097241296869580334779 3}{2^7\,3^{18}\,5^6\,7^5\,11^3\,13^5\,17^2\,19^2\,23^2}$$

$$D_1^5(8) = 1/9^5 = 1/3^{10}, \qquad D_1^5(26) = 1/27^5 = 1/3^{15}.$$

We **must** provide at least $\mathbf{1/3^3 = 1/27}$ at OW5:

OW	5	4	3	2	1
$P^{(3)}_{qg}\big\|_{C_A n_f^3}$	$1/27$	$1/81$	$1/243$	$1/729$	$1/2187$

Hardest Singlet Case

$P^{(3)}_{qg}\big|_{C_A n_f^3}$: Basis with **125** unknown integer coefficients.

Computed $N = \mathbf{2}, \ldots, \mathbf{46}$, insufficient to determine a "good" solution.

Moment calculations become very computationally demanding.
Hardest single diagram computed at $N = \mathbf{46}$,

- $\sim$ **2** weeks wall-time [16 cores, 192GB RAM, 24TB scratch space]
- $\sim$ **13TB** peak disk usage by **`TFORM`**

$\longrightarrow$ **no more moments!** ($N \to N + \mathbf{2}$: resource req. $\sim$ double)

We need to make the reconstruction basis smaller.

Use additional constraints/assumptions:

- large-N limit constants: ζ_i only (no $\mathbf{ln\,2}, \mathbf{Li_4(1/2)}, \mathbf{Li_5(1/2)}$) ($\to$ **124**)
- $\#S_{1,2} = -\#S_{2,1}$ ($\to$ **117**)

117 unknowns. Solution with $N = \mathbf{2}, \ldots, \mathbf{44}$, $N = \mathbf{46}$ checks.

Non-singlet Splitting Functions

n_f^3 terms of $P_{ns}^{(3),\pm}$ are already known to all orders in a_s. [Gracey '94]

Here we determine the n_f^2 terms of $P_{ns}^{(3),+}$ (even N) and $P_{ns}^{(3),-}$ (odd N).
Colour factors to determine:

- $C_F^2 n_f^2$
- $C_A C_F n_f^2$ – diagrams are **very hard** to compute at higher N values!

Method: **decompose in two ways**,

$$
\begin{aligned}
P_{ns}^{(3),\pm}\{n_f^2\{C_F^2, C_A C_F\}\} &= n_f^2\left(2C_F^2 A + C_F(C_A - 2C_F)B^\pm\right) \\
&= n_f^2\left(2C_F^2(A - B^\pm) + C_F C_A B^\pm\right).
\end{aligned}
$$

A should be **common to both $P_{ns}^\pm$**; use **both odd and even N**. Large n_c.

Compute (easier) $C_F^2 n_f^2$ diagrams to higher N to determine $(A - B^\pm)$.

From these, determine B^+ and B^- and hence $P_{ns}^{(3),+}$ and $P_{ns}^{(3),-}$.

Non-singlet Splitting Functions

$2C_F^2 n_f^2 (A - B^{\pm})$: Initial guess, **139** basis elements (incl. $SW5$). Too big!

- Enforce large-N limit $\sim \ln N$ (no $\ln^2 N$ etc) ($\rightarrow$ **123**)
- Large-N constants: ζ_i only ($\rightarrow$ **119**)
- $\# S_{1,2} = \# S_{2,1}$ ($\rightarrow$ **115**)

Solution for $(A - B^{+})$ with $N = 2, \ldots, 40$. $N = 42$ checks.
Solution for $(A - B^{-})$ with $N = 3, \ldots, 37$. $N = 39$ checks.

$2C_F^2 n_f^2 A$: basis as above, no alternating sums. **65** elements. **Still** too big!

- No "many-index" sums. Discard $S_{1,1,2}, S_{1,2,1}, S_{2,1,1}$ and $S_{1,1,1,2}, S_{1,1,2,1}, S_{1,2,1,1}, S_{2,1,1,1}, S_{1,2,2}, S_{2,1,2}, S_{2,2,1}$. ($\rightarrow$ **54**)
  (We see this at 3 loops, and in 4 loop singlet)

Solution with $N = 2, 3, \ldots, 17$, and $N = 18, 19, \ldots, 22$ check.

VERIFICATION

Check against existing literature:

- Linear comb. of n_f^3 terms of $P_{qq}^{(3)}$, $P_{gq}^{(3)}$, and $P_{gq}^{(3)}$, $P_{gg}^{(3)}$ ✓ [Gracey '96,'98]
- Large-N prediction of P_{ns} [Dokshitzer,Marchesini,Salam '06]

  If $P_{ns}^{(i-1)} = -A_q^i \ln N + B_q^i - C_q^i \ln N/N + \mathcal{O}(1/N^2)$, we have that

  $C_q^1 = 0, \;\; C_q^2 = (A_q^1)^2, \;\; C_q^3 = 2A_q^1 A_q^2 : \;\; C_q^4 = (A_q^2)^2 + 2A_q^1 A_q^3.$

  Here: $C_q^4 = \frac{1216}{81} C_F^2 n_f^2 + \mathcal{O}(n_f)$. ✓
- Small-x ($x \to 0$) Double Log. Resummations [(Davies,)Kom,Vogt '12('XX)]

  $P_{ij}^{(n)} \sim \frac{1}{x}(\ln^{n-1} x + \cdots + con.) + x^0(\ln^{2n} x + \cdots + con.) + \cdots$

  $x^{0,\text{even}}(\ln^{2n} x + \ln^{2n-1} x + \ln^{2n-2} x)$ known to "all orders" in a_s.

  In agreement with fixed order $P_{ij}^{(3)}$ (and $P_{ns}^{(3),\pm}$) computed here. ✓

Verification

- Large-x ($x \to 1$) Double Log. Resummations [Soar,Moch,Vermaseren,Vogt '10]

$$\frac{d}{d\ln Q^2}F = \frac{d}{d\ln Q^2}(C\,q) = \left(\beta\frac{dC}{da_s} + C\,P\right)q = \underbrace{\left[\left(\beta\frac{dC}{da_s} + C\,P\right)C^{-1}\right]}_{K}F.$$

K: *Physical Kernel*. Conjecture: *single-log. enhanced to all a_s orders*.

$\Longrightarrow$ cancellation between double logs of C, P.

$\longrightarrow$ prediction of $P \sim a_s^4(\ln^6(1-x) + \ln^5(1-x) + \ln^4(1-x))$ ✓

- Cusp Anomalous Dimension at a_s^4: given by A in large-N limit ✓
  [Henn,Lee,Smirnov,Smirnov,Steinhauser '16] [Grozin '16] [Lee,Smirnov,Smirnov,Steinhauser '17]

CURRENT STATUS OF ALL-N EXPRESSIONS

Discussed here:

- n_f^3 terms of $P_{qq}^{(3)}$, $P_{qg}^{(3)}$, $P_{gq}^{(3)}$, $P_{gg}^{(3)}$
- n_f^2 terms of $P_{ns}^{(3),\pm}$

Also completed:

- n_f^2 terms of $P_V^{(3)}$
- all n_f powers, n_f^0: large-n_c limit of A [Moch,Ruijl,Ueda,Vermaseren,Vogt '17]
- all n_f powers, n_f^0: ζ_5, ζ_4 terms of $P_{ns}^{(3),\pm}$
- all n_f powers : ζ_3 terms of $P_{ns}^{(3),\pm}$

What now? Numerical Approximations

We have analytic expressions for some (large-n_f/large-n_c) colour factors.

What about the remaining parts? [Moch,Ruijl,Ueda,Vermaseren,Vogt '17]

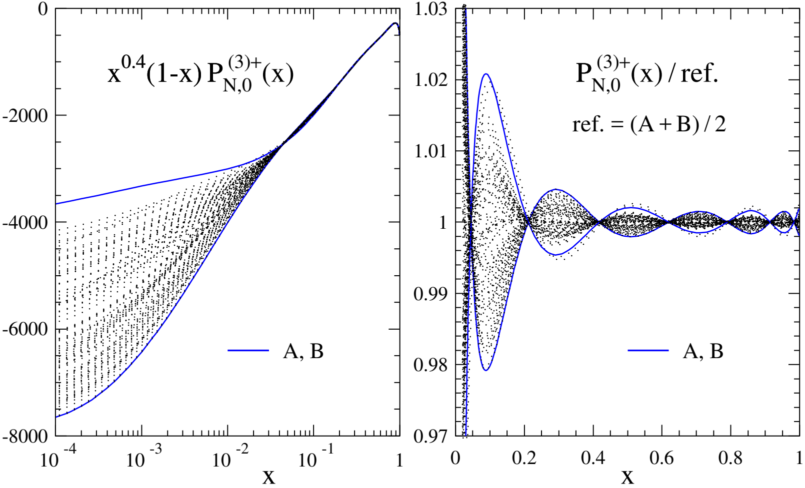
- numerical approximations using the Mellin moments
- phenomenologically useful, if approximate, results.

Choose ansatz for fit. E.g. for $P_{ns}^{(4),+}$ n_f^0 and n_f^1 terms,

- Large-x:
  - A_q^4, B_q^4 (coeffs. of $\ln N$ and *const* in N-space)
  - 2 of 3 suppressed logs: $(1-x)\ln^k(1-x)$ $(k=1,2,3)$
- Small-x:
  - 2 of 3 unknown logs: $\ln^k x$ $(k=1,2,3)$
- Interpolation:
  - 1 of 10 2-parameter polynomials in x

$\rightarrow$ Family of 90 trial functions. Parameters set using 8 known moments.

n_f^0 FIT

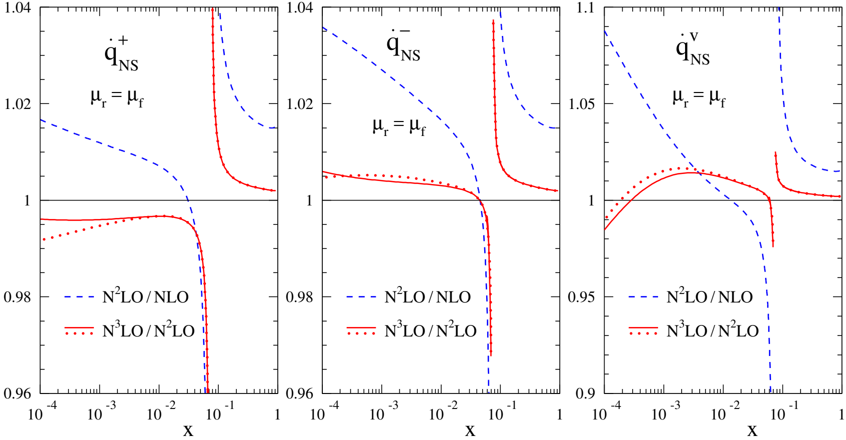


$\boldsymbol{A}, \boldsymbol{B}$ approximations bracket the trial functions.

Coefficient $\boldsymbol{x^{0.4}(1-x)}$: for display. [Moch,Ruijl,Ueda,Vermaseren,Vogt '17]

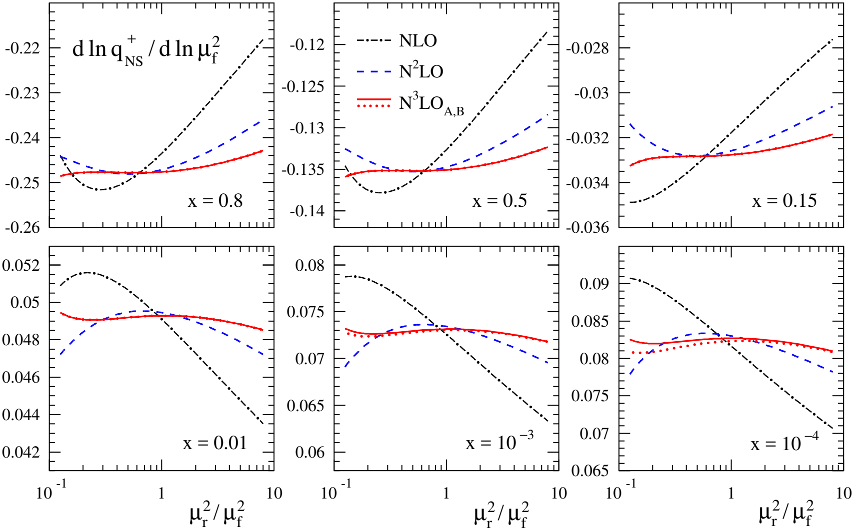
PDF Evolution

$$\dot{q}_{NS}^{\pm,V} = \frac{d}{d\ln\mu^2} q_{NS}^{\pm,V}$$



Model PDF: $xq_{ns}^{\pm,v} = x^{0.5}(1-x)^3$, $\alpha_s = 0.2$. [Moch,Ruijl,Ueda,Vermaseren,Vogt '17]

Scale Dependence



[Moch,Ruijl,Ueda,Vermaseren,Vogt '17]

THE "NO-π^2 CONJECTURE"

An old observation:

- Euclidean physical quantities have no ζ_4 $(= \pi^4/90)$ terms
- E.g. 5-loop corrections to the Adler function

Broken by 5-loop corrections to scalar–quark, scalar–gluon correlators.

C-scheme introduced. For $C = 0$: [Boito,Jamin,Miravitllas '16]

$$a_s = \bar{a}_s + \left(\frac{\beta_2}{\beta_0} - \frac{\beta_1^2}{\beta_0^2}\right)\bar{a}_s^3 + \left(\frac{\beta_3}{2\beta_0} - \frac{\beta_1^3}{2\beta_0^3}\right)\bar{a}_s^4 + \left(\frac{\beta_4}{3\beta_0} - \frac{\beta_1\beta_3}{6\beta_0^2} + \frac{5\beta_2^2}{3\beta_0^2} - \frac{3\beta_1^2\beta_2}{\beta_0^3} + \frac{7\beta_1^4}{6\beta_0^4}\right)\bar{a}_s^5 + \mathcal{O}(\bar{a}_s^6).$$

Conjecture: [Jamin,Miravitllas '17]

- *in this scheme, Euclidean physical quantities do not have even-n ζ_n terms.*

Verification:

- physical quantities built from scalar–quark, scalar–gluon correlators have no ζ_4, ζ_6 in their 5-loop corrections in this scheme.

“NO-π^2” IN DIS?

Euclidean physical quantities, **Physical Kernels**: [Davies,Vogt ‘17]

$$\frac{d}{d\ln Q^2}F = \left[\left(\beta\frac{dC}{da_s} + C\,P\right)C^{-1}\right]F = K\,F = \left[\sum_{i=0}^{\infty} a_s^{i+1}K^{(i)}\right]F.$$

The relevant terms at a_s^4: $\left[\widetilde{f} = f|_{\zeta_4}\right]$

$$\widetilde{K}_{2,ns}^{(3)} = \widetilde{P}_{ns}^{(3),+} - 3\beta_0\,\tilde{c}_{2,ns}^{(3)}\ , \qquad \widetilde{K}_3^{(3)} = \widetilde{P}_{ns}^{(3),-} - 3\beta_0\,\tilde{c}_3^{(3)}\ ,$$

(F_2, F_ϕ) system: $\begin{pmatrix} \widetilde{K}_{22}^{(3)} & \widetilde{K}_{2\phi}^{(3)} \\ \widetilde{K}_{\phi 2}^{(3)} & \widetilde{K}_{\phi\phi}^{(3)} \end{pmatrix}$ $\left[\text{dep. on} \begin{pmatrix} \widetilde{P}_{qq}^{(3)} & \widetilde{P}_{qg}^{(3)} \\ \widetilde{P}_{gq}^{(3)} & \widetilde{P}_{gg}^{(3)} \end{pmatrix}\right]$

$\widetilde{P}_{ns}^{(3),\pm}$ are known analytically (reconstructed from moments: see above)

- $\widetilde{K}_{2,ns}^{(3)}(N) = \widetilde{K}_3^{(3)}(N) = 0$. ✓

$\widetilde{P}_{ij}^{(3)}$ are known only at $N = 2, 4$

- $\widetilde{K}_{ab}^{(3)}(N = \{2, 4\}) = 0$. ✓ $\Longrightarrow$ pred. of $\widetilde{P}_{ij}^{(3)}(N)$.

"NO-π^2" IN DIS?

The relevant terms at a_s^5: $[a = 2, ns,\ 3 \mid \sigma = +, -]$ $\left[\widehat{f} = f|_{\zeta_6}\right]$

$$\widehat{K}_a^{(4)} = \widehat{P}_{ns}^{(4),\sigma} - 4\beta_0\, \hat{c}_a^{(4)}$$

$$\widetilde{K}_a^{(4)} = \widetilde{P}_{ns}^{(4),\sigma} - 3\beta_1\, \tilde{c}_a^{(3)} - 4\beta_0 \left(\tilde{c}_a^{(4)} - c_a^{(1)}\, \tilde{c}_a^{(3)}\right)$$

$c_a^{(4)}$ known at $N \leq 6$, $P_{ns}^{(4),\sigma}$ known at $N = 2, 3$

- $\widehat{K}_{2,ns}^{(4)}(N = 2) = 0$, $\widehat{K}_3^{(4)}(N = 3) = 0$. ✓ $\Longrightarrow$ pred. of $\widehat{P}_{ns}^{(4),\sigma}(N \leq 6)$
- $\widetilde{K}_{2,ns}^{(4)}(N = 2) = 0$, $\widetilde{K}_3^{(4)}(N = 3) = 0$, only after scheme trf.! ✓
  $\Longrightarrow$ pred. of $\widetilde{P}_{ns}^{(4),\sigma}(N \leq 6)$

SUMMARY

FORCER allows us to determine moments of **4-loop Splitting Functions**.

Using these one can:

- Reconstruct analytic expressions for the "easier" colour factors
  - large-n_f and large-n_c parts
  - non-singlet: all colour factors of "higher-ζ_n" terms
- Create numerical approximations for the remaining colour factors
  - perturbative expansion of Splitting Functions well converging
  - reduced scale dependence

DIS provides additional verification of the "no-π^2 conjecture":

- if you believe it, yields predictions of currently unknown quantities
  - analytic expressions at a_s^4
  - Mellin moments at a_s^5

Backup: Valence Splitting Function

We have not discussed P_V. Consider $P_S = P_V - P_{ns}^{-}$.

It contains terms with cubic Casimir $d^{abc}d_{abc}/n_c\, n_f^2$.

We have **just 12** moments: $N = 3, \ldots, 25$. ($1, 27$ as a check)

Conjecture (DMS):

$$P_S^{(3)} = -\frac{2}{3} n_f \frac{d}{dN} P_S^{(2)} + \text{RR terms}$$

Reciprocity Respecting terms: $f(x) = x f(1/x)$ in x-space.

RR basis, no many-index sums, no $N = 1$ ζ_i: 59 unknown coefficients.

Solution!